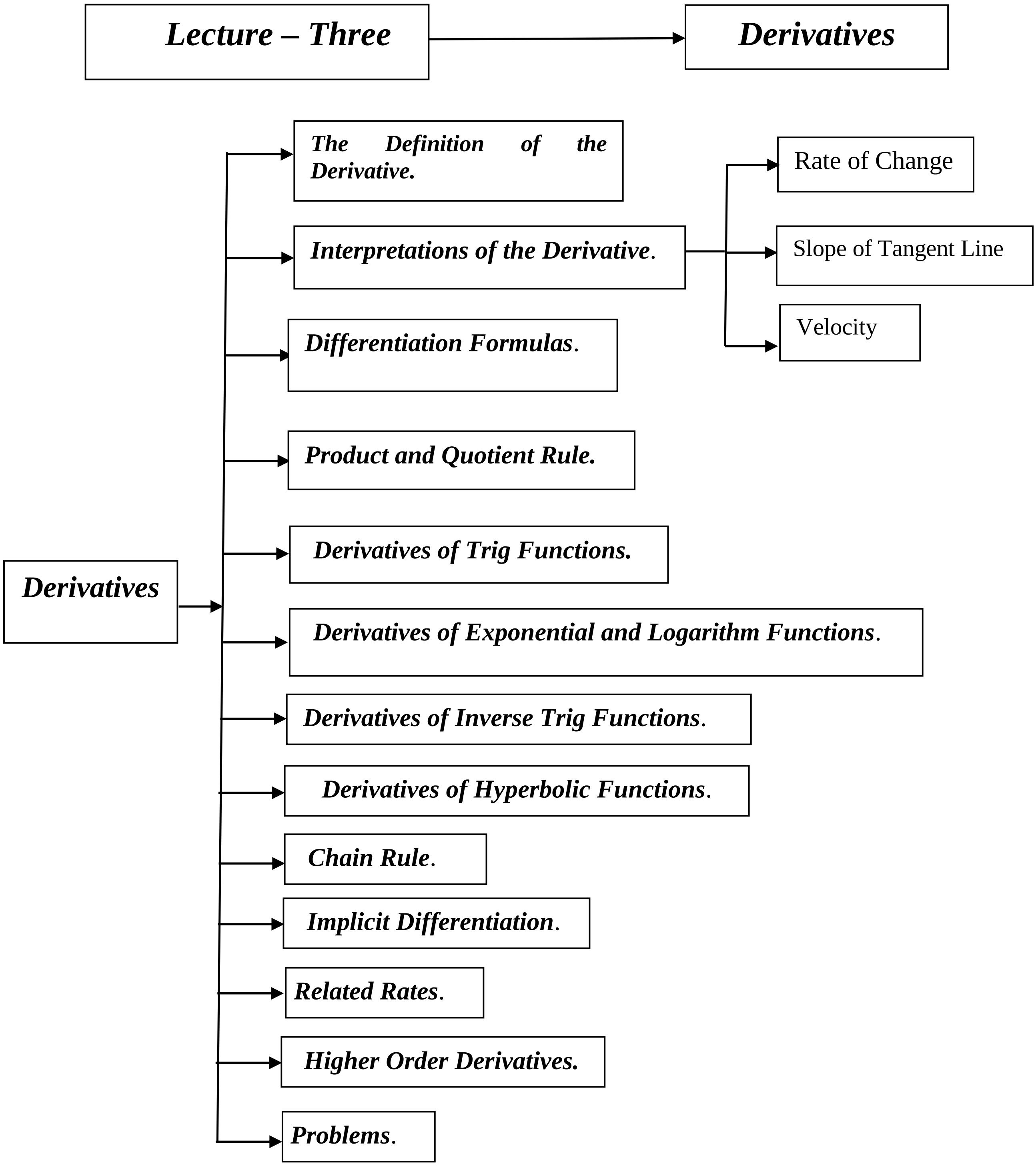
Calculus – I

First year/ 1st Semester - Chemical and Petroleum Engineering Department

By

Dr. Mustafa B. Al-hadithi



Derivatives

1- *The Definition of the Derivative.*

The **derivative of** $f(x)$ **with respect to** x is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We often "read" $f'(x)$ as "f prime of x".

Example 1 Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

First plug the function into the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h}$$

We can't just plug in ***h=0*** since this will give us a division by zero error. So we are going to have to do some work.

$$f'(x) = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 - 16h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h(4x + 2h - 16)}{h}$$

$$= \lim_{h \to 0} 4x + 2h - 16$$

$$= 4x - 16$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Example 2 Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution.

$$g'(t)=\lim_{h\to 0}\frac{g(t+h)-g(t)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{t+h}{t+h+1}-\frac{t}{t+1}\right)$$
$$g'(t)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(t+h)(t+1)-t(t+h+1)}{(t+h+1)(t+1)}\right)$$
$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{t^2+t+th+h-(t^2+th+t)}{(t+h+1)(t+1)}\right)$$
$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right)$$

After the simplification we only have terms with *h*'s in them left in the numerator and so we can now cancel an *h* out. So, upon canceling the *h* we can evaluate the limit and get the derivative.

$$g'(t)=\lim_{h\to 0}\frac{1}{(t+h+1)(t+1)}$$
$$=\frac{1}{(t+1)(t+1)}$$
$$=\frac{1}{(t+1)^2}$$

The derivative is then,

$$g'(t)=\frac{1}{(t+1)^2}$$

Example 3 Find the derivative of the following function using the definition of the derivative.

$$R(z)=\sqrt{5z-8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$R'(z)=\lim_{h\to 0}\frac{R(z+h)-R(z)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{5(z+h)-8}-\sqrt{5z-8}}{h}$$

We multiply both the numerator and denominator by the numerator except we change the sign between the two terms.

$$R'(z) = \lim_{h \to 0} \frac{\left(\sqrt{5(z+h)-8} - \sqrt{5z-8}\right)}{h} \frac{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h \to 0} \frac{5z + 5h - 8 - (5z - 8)}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h \to 0} \frac{5h}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

Again, after the simplification we have only h's left in the numerator. So, cancel the h and evaluate the limit.

$$R'(z) = \lim_{h \to 0} \frac{5}{\sqrt{5(z+h)-8} + \sqrt{5z-8}}$$

$$= \frac{5}{\sqrt{5z-8} + \sqrt{5z-8}}$$

$$= \frac{5}{2\sqrt{5z-8}}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Example 4 Determine $f'(0)$ for $f(x) = |x|$

So, plug into the definition and simplify.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

We will have to look at the two one sided limits and recall that

$$|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\begin{aligned}\lim_{h\to 0^-}\frac{|h|}{h} &= \lim_{h\to 0^-}\frac{-h}{h} && \text{because } h<0 \text{ in a left-hand limit.}\\ &= \lim_{h\to 0^-}(-1)\\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0^+}\frac{|h|}{h} &= \lim_{h\to 0^+}\frac{h}{h} && \text{because } h>0 \text{ in a right-hand limit.}\\ &= \lim_{h\to 0^+}1\\ &= 1\end{aligned}$$

The two one-sided limits are different and so

$$\lim_{h\to 0}\frac{|h|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

The preceding discussion leads to the following definition.

Definition

A function $f(x)$ is called **differentiable** at $x=a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

Theorem

If $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$.

Given a function $y=f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to x.

$$f'(x)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}\big(f(x)\big)=\frac{d}{dx}(y)$$

2- ***Interpretations of the Derivative.***

a- Rate of Change

$f(x)$ represents a quantity at any x then the derivative $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x=a$.

Example 1 Suppose that the amount of water in a holding tank at t minutes is given by $V(t)=2t^2-16t+35$. Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at $t=1$ minute?

(b) Is the volume of water in the tank increasing or decreasing at $t=5$ minutes?

Is the volume of water in the tank ever not changing? If so, when?

The derivative is.

$$V'(t) = 4t - 16 \qquad \text{OR} \qquad \frac{dV}{dt} = 4t - 16$$

If the rate of change was positive then the quantity was increasing and if the rate of change was negative then the quantity was decreasing.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

In this case all that we need is the rate of change of the volume at $t = 1$ or,

$$V'(1) = -12 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=1} = -12$$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

Again, we will need the rate of change at $t = 5$.

$$V'(5) = 4 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at $t = 5$.

The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$V'(t) = 0 \qquad \text{OR} \qquad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \qquad \Rightarrow \qquad t = 4$$

So at $t = 4$ the volume isn't changing. Note that all this is saying is that for a brief instant the volume isn't changing. It doesn't say that at this point the volume will quit changing permanently.

b- <u>Slope of Tangent Line.</u>

This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x = a$ is $f'(a)$. The tangent line then is given by,

$$y = f(a) + f'(a)(x - a)$$

Example 2 Find the tangent line to the following function at $z = 3$.

$$R(z) = \sqrt{5z - 8}$$

Solution

We first need the derivative of the function and we found that in Example 3 in the last section. The derivative is,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Now all that we need is the function value and derivative (for the slope) at $z = 3$.

$$R(3) = \sqrt{7} \qquad\qquad m = R'(3) = \frac{5}{2\sqrt{7}}$$

The tangent line is then,

$$y = \sqrt{7} + \frac{5}{2\sqrt{7}}(z-3)$$

c- Velocity.

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after t units of time the velocity of the object at $t = a$ is given by $f'(a)$.

Example 3 Suppose that the position of an object after t hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at $t = 10$ hours?

(b) Does the object ever stop moving?

The derivative is,

$$g'(t) = \frac{1}{(t+1)^2}$$

(a) Is the object moving to the right or the left at $t = 10$ hours?

To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at $t = 10$.

$$g'(10) = \frac{1}{121}$$

So the velocity at $t = 10$ is positive and so the object is moving to the right at $t = 10$.

(b) Does the object ever stop moving?

The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can't be zero.

Therefore, the object will never stop moving.

3- ***Differentiation Formulas.***

We will start in this section with some of the basic properties and formulas. We will give the properties and formulas in this section in both "prime" notation and "fraction" notation.

1) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$ **OR** $\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$

2) $(cf(x))' = cf'(x)$ **OR** $\frac{d}{dx}(cf(x)) = c\frac{df}{dx}$, c is any number

Next, let's take a quick look at a couple of basic "computation" formulas that will allow us to actually compute some derivatives.

Formulas

1) If $f(x) = c$ then $f'(x) = 0$ **OR** $\frac{d}{dx}(c) = 0$

2) If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ **OR** $\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number.

This formula is sometimes called the **power rule**.

Example 1 Differentiate each of the following functions.

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$ **(b)** $g(t) = 2t^6 + 7t^{-6}$

(C) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$

Solution

(a) $f(x) = 15x^{100} - 3x^{12} + 5x - 46$

$$f'(x) = 15(100)x^{99} - 3(12)x^{11} + 5(1)x^0 - 0$$
$$= 1500x^{99} - 36x^{11} + 5$$

(b) $g(t) = 2t^6 + 7t^{-6}$

$$g'(t) = 2(6)t^5 + 7(-6)t^{-7}$$
$$= 12t^5 - 42t^{-7}$$

(C) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \frac{2}{\sqrt[5]{x^2}}$

$$T(x) = x^{\frac{1}{2}} + 9\left(x^7\right)^{\frac{1}{3}} - \frac{2}{\left(x^2\right)^{\frac{1}{5}}}$$

$$= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}}$$

$$= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}}$$

$$T'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} + \frac{63}{3}x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}}$$

Example 2 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

Solution

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

We can simplify this rational expression however as follows.

$$h(t)=\frac{2t^5}{t^2}+\frac{t^2}{t^2}-\frac{5}{t^2}=2t^3+1-5t^{-2}$$

This is a function that we can differentiate.

$$h'(t)=6t^2+10t^{-3}$$

4- ***Product and Quotient Rule.***

To differentiate products and quotients we have the **Product Rule** and the **Quotient Rule**.

Product Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$(f\,g)'=f'\,g+f\,g'$$

Quotient Rule

If the two functions *f(x)* and *g(x)* are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)'=\frac{f'\,g-f\,g'}{g^2}$$

Example 1 Differentiate each of the following functions.

(a) $y=\sqrt[3]{x^2}\left(2x-x^2\right)$

(b) $f(x)=\left(6x^3-x\right)\left(10-20x\right)$

Solution

(a) $y=\sqrt[3]{x^2}\left(2x-x^2\right)$

$$y=x^{\frac{2}{3}}\left(2x-x^2\right)$$

$$y'=\frac{2}{3}x^{-\frac{1}{3}}\left(2x-x^2\right)+x^{\frac{2}{3}}\left(2-2x\right)$$

However, with some simplification we can arrive to.

$$y'=\frac{4}{3}x^{\frac{2}{3}}-\frac{2}{3}x^{\frac{5}{3}}+2x^{\frac{2}{3}}-2x^{\frac{5}{3}}=\frac{10}{3}x^{\frac{2}{3}}-\frac{8}{3}x^{\frac{5}{3}}$$

(b) $f(x) = (6x^3 - x)(10 - 20x)$

This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'(x) &= (18x^2 - 1)(10 - 20x) + (6x^3 - x)(-20) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Example 2 Differentiate each of the following functions.

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Solution

(a) $W(z) = \dfrac{3z + 9}{2 - z}$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'(z) &= \frac{3(2 - z) - (3z + 9)(-1)}{(2 - z)^2} \\ &= \frac{15}{(2 - z)^2} \end{aligned}$$

(b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$\begin{aligned} h'(x) &= \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}(x^2 - 2) - 4x^{\frac{1}{2}}(2x)}{(x^2 - 2)^2} \\ &= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{(x^2 - 2)^2} \\ &= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{(x^2 - 2)^2} \end{aligned}$$

Example 3 Suppose that the amount of air in a balloon at any time t is given by

$$V(t) = \frac{6\sqrt[3]{t}}{4t + 1}$$

Determine if the balloon is being filled with air or being drained of air at $t = 8$.

Solution

This will require the quotient rule.

$$V'(t) = \frac{2t^{-\frac{2}{3}}(4t+1) - 6t^{\frac{1}{3}}(4)}{(4t+1)^2}$$

$$= \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t+1)^2}$$

$$= \frac{-16t^{\frac{1}{3}} + \frac{2}{t^{\frac{2}{3}}}}{(4t+1)^2}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate. The rate of change of the volume at t=8 is then,

$$V'(8) = \frac{-16(2) + \frac{2}{4}}{(33)^2} \qquad (8)^{\frac{1}{3}} = 2 \qquad (8)^{\frac{2}{3}} = \left((8)^{\frac{1}{3}}\right)^2 = (2)^2 = 4$$

$$= -\frac{63}{2178} = -\frac{7}{242}$$

So, the rate of change of the volume at $t = 8$ is negative and so the volume must be decreasing. Therefore air is being drained out of the balloon at $t = 8$.

As a final topic let's note that the product rule can be extended to more than two functions, for instance.

$$(f\,g\,h)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$

$$(f\,g\,h\,w)' = f'\,g\,h\,w + f\,g'\,h\,w + f\,g\,h'\,w + f\,g\,h\,w'$$

5- ***Derivatives of Trig Functions***.

Fact

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \qquad\qquad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

Example 1 Evaluate each of the following limits.

(a) $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{6\theta}$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(6x)}{x}$

(c) $\displaystyle\lim_{x\to 0}\frac{x}{\sin(7x)}$

(d) $\displaystyle\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}$

(e) $\displaystyle\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$

(f) $\displaystyle\lim_{z\to 0}\frac{\cos(2z)-1}{z}$

Solution

(a) $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{6\theta}$

$$\lim_{\theta\to 0}\frac{\sin\theta}{6\theta}=\frac{1}{6}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{1}{6}(1)=\frac{1}{6}$$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(6x)}{x}$

Doing the change of variables on this limit gives,

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(6x)}{x}&=6\lim_{x\to 0}\frac{\sin(6x)}{6x} \qquad\qquad \text{let } \theta=6x\\ &=6\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\\ &=6(1)\\ &=6\end{aligned}$$

(c) $\lim\limits_{x \to 0} \dfrac{x}{\sin(7x)}$

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$

$$\lim_{x \to 0} \frac{x}{\sin(7x)} = \lim_{x \to 0} \frac{1}{\frac{\sin(7x)}{x}}$$

$$= \frac{\lim\limits_{x \to 0} 1}{\lim\limits_{x \to 0} \frac{\sin(7x)}{x}}$$

$$= \frac{1}{\lim\limits_{x \to 0} \frac{\sin(7x)}{x}}$$

$$\lim_{x \to 0} \frac{x}{\sin(7x)} = \frac{1}{\lim\limits_{x \to 0} \frac{7\sin(7x)}{7x}}$$

$$= \frac{1}{7\lim\limits_{x \to 0} \frac{\sin(7x)}{7x}}$$

$$= \frac{1}{(7)(1)}$$

$$= \frac{1}{7}$$

(d) $\lim\limits_{t \to 0} \dfrac{\sin(3t)}{\sin(8t)}$

$$\lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)} = \lim_{t \to 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)}$$

Now, the fact wants a *t* in the denominator of the first and in the numerator of the second. This is easy enough to do if we multiply the whole thing by t/t (which is just one after all and so won't change the problem) and then do a little rearranging as follows

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}\frac{t}{t}\\ &= \lim_{t\to 0}\frac{\sin(3t)}{t}\frac{t}{\sin(8t)}\\ &= \left(\lim_{t\to 0}\frac{\sin(3t)}{t}\right)\left(\lim_{t\to 0}\frac{t}{\sin(8t)}\right)\end{aligned}$$

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \left(\lim_{t\to 0}\frac{3\sin(3t)}{3t}\right)\left(\lim_{t\to 0}\frac{8t}{8\sin(8t)}\right)\\ &= \left(3\lim_{t\to 0}\frac{\sin(3t)}{3t}\right)\left(\frac{1}{8}\lim_{t\to 0}\frac{8t}{\sin(8t)}\right)\\ &= (3)\left(\frac{1}{8}\right)\\ &= \frac{3}{8}\end{aligned}$$

(e) $\lim\limits_{x\to 4}\dfrac{\sin(x-4)}{x-4}$

So, let $\theta = x-4$ and then notice that as $x \to 4$ we have $\theta \to 0$. Therefore, after doing the change of variable the limit becomes,

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

(f) $\lim\limits_{z\to 0}\dfrac{\cos(2z)-1}{z}$

$$\begin{aligned}\lim_{z\to 0}\frac{\cos(2z)-1}{z} &= \lim_{z\to 0}\frac{2(\cos(2z)-1)}{2z}\\ &= 2\lim_{z\to 0}\frac{\cos(2z)-1}{2z}\\ &= 2(0)\\ &0\end{aligned}$$

We'll start with finding the derivative of the sine function. To do this we will need to use the definition of the derivative. It's been a while since we've had to use this, but sometimes there just isn't anything we can do about it. Here is the definition of the derivative for the sine function.

$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

Since we can't just plug in $h = 0$ to evaluate the limit we will need to use the following trig formula on the first sine in the numerator.

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Doing this gives us,

$$\begin{aligned}\frac{d}{dx}(\sin(x)) &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}\\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}\\ \frac{d}{dx}(\sin(x)) &= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\end{aligned}$$

At this point all we need to do is use the limits in the fact above to finish out this problem.

$$\frac{d}{dx}(\sin(x)) = \sin(x)(0) + \cos(x)(1) = \cos(x)$$

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Let's take a look at tangent. Tangent is defined as,

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let's do that.

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)\\ &= \frac{\cos(x)\cos(x) - \sin(x)\left(-\sin(x)\right)}{\left(\cos(x)\right)^2}\\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}\end{aligned}$$

Now, recall that $\cos^2(x) + \sin^2(x) = 1$ and if we also recall the definition of secant in terms of cosine we arrive at,

$$\begin{aligned}\frac{d}{dx}\left(\tan(x)\right) &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x)\end{aligned}$$

The remaining three trig functions are also quotients involving sine and/or cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

Derivatives of the six trig functions

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$$

Example 2 Differentiate each of the following functions.

(a) $g(x) = 3\sec(x) - 10\cot(x)$

(b) $h(w) = 3w^{-4} - w^2\tan(w)$

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

Solution

(a) $g(x) = 3\sec(x) - 10\cot(x)$

$$g'(x) = 3\sec(x)\tan(x) - 10\left(-\csc^2(x)\right)$$
$$= 3\sec(x)\tan(x) + 10\csc^2(x)$$

(b) $h(w) = 3w^{-4} - w^2\tan(w)$

$$h'(w) = -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right)$$
$$= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

$$y' = 5\cos(x)\cos(x) + 5\sin(x)\left(-\sin(x)\right) - 4\csc(x)\cot(x)$$
$$= 5\cos^2(x) - 5\sin^2(x) - 4\csc(x)\cot(x)$$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

$$P'(t) = \frac{\cos(t)\left(3 - 2\cos(t)\right) - \sin(t)\left(2\sin(t)\right)}{\left(3 - 2\cos(t)\right)^2}$$
$$= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{\left(3 - 2\cos(t)\right)^2}$$

$$P'(t) = \frac{3\cos(t) - 2\left(\cos^2(t) + \sin^2(t)\right)}{\left(3 - 2\cos(t)\right)^2}$$
$$= \frac{3\cos(t) - 2}{\left(3 - 2\cos(t)\right)^2}$$

6- ***Derivatives of Exponential and Logarithm Functions.***

The most common exponential and logarithm functions in a calculus course are the natural exponential function, $\mathbf{e}^x$, and the natural logarithm function, ln(x) . We will take a more general approach however and look at the general exponential and logarithm function.

a- ***Exponential Functions.***

We'll start off by looking at the exponential function,

$$f(x) = a^x$$

We're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h} \end{aligned}$$

Now let's notice that the limit we've got above is exactly the definition of the derivative of $f(x) = a^x$ at $x = 0$, *i.e.* $f'(0)$. Therefore, the derivative becomes,

$$f'(x) = f'(0)\,a^x$$

e is the unique positive number for which $\lim\limits_{h\to 0}\dfrac{\mathbf{e}^h - 1}{h} = 1$

Fact 1

For the natural exponential function, $f(x) = \mathbf{e}^x$ we have $f'(0) = \lim\limits_{h\to 0}\dfrac{\mathbf{e}^h-1}{h} = 1$.

So, provided we are using the natural exponential function we get the following.

$$f(x) = \mathbf{e}^x \qquad \Rightarrow \qquad f'(x) = \mathbf{e}^x$$

$$f(x) = a^x \qquad \Rightarrow \qquad f'(x) = a^x \ln(a)$$

b- ***Logarithm Functions.***

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad x > 0$$

$$\frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x} \qquad x\neq 0$$

Using the change of base formula we can write a general logarithm as,

$$\log_a x=\frac{\ln x}{\ln a}$$

Differentiation is then fairly simple.

$$\begin{aligned}\frac{d}{dx}\left(\log_a x\right)&=\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\&=\frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right)\\&=\frac{1}{x\ln a}\end{aligned}$$

Here is a summary of the derivatives in this section.

$\frac{d}{dx}\left(\mathbf{e}^x\right)=\mathbf{e}^x$	$\frac{d}{dx}\left(a^x\right)=a^x\ln a$
$\frac{d}{dx}\left(\ln x\right)=\frac{1}{x}$	$\frac{d}{dx}\left(\log_a x\right)=\frac{1}{x\ln a}$

Example 1 Differentiate each of the following functions.

(a) $R(w)=4^w-5\log_9 w$

(b) $f(x)=3\mathbf{e}^x+10x^3\ln x$

(c) $y=\dfrac{5\mathbf{e}^x}{3\mathbf{e}^x+1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w)=4^w\ln 4-\frac{5}{w\ln 9}$$

(b) Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned}f'(x)&=3\mathbf{e}^x+30x^2\ln x+10x^3\left(\frac{1}{x}\right)\\&=3\mathbf{e}^x+30x^2\ln x+10x^2\end{aligned}$$

(c) We'll need to use the quotient rule on this one.

$$y' = \frac{5\mathbf{e}^x\left(3\mathbf{e}^x+1\right)-\left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x+1\right)^2}$$

$$= \frac{15\mathbf{e}^{2x}+5\mathbf{e}^x-15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x+1\right)^2}$$

$$= \frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x+1\right)^2}$$

7- ***Derivatives of Inverse Trig Functions***.

If *f(x)* and *g(x)* are inverse functions then,

$$g'(x) = \frac{1}{f'\left(g(x)\right)}$$

Recall as well that two functions are inverses if $f\left(g(x)\right) = x$ and $g\left(f(x)\right) = x$.

a- ***Inverse Sine.***

$y = \sin^{-1} x \qquad \Leftrightarrow \qquad \sin y = x \qquad \text{for} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Note as well that since $-1 \le \sin(y) \le 1$ we also have $-1 \le x \le 1$.

Example 1 Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

Solution

So we are really asking what angle y solves the following equation.

$$\sin(y) = \frac{1}{2}$$

and we are restricted to the values of y above.

From a unit circle we can quickly see that $y = \frac{\pi}{6}$.

We have the following relationship between the inverse sine function and the sine function.

$$\sin\left(\sin^{-1} x\right) = x \qquad\qquad \sin^{-1}\left(\sin x\right) = x$$

Let's start by recalling the definition of the inverse sine function.

$$y = \sin^{-1}(x) \qquad \Rightarrow \qquad x = \sin(y)$$

$$\cos\left(\sin^{-1} x\right) = \cos(y)$$

Now, recall that

$$\cos^2 y + \sin^2 y = 1 \qquad \Rightarrow \qquad \cos y = \sqrt{1-\sin^2 y}$$

Using this, the denominator is now,

$$\cos\left(\sin^{-1} x\right) = \cos(y) = \sqrt{1-\sin^2 y}$$

Now, use the second part of the definition of the inverse sine function. The denominator is then,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

This means that we can use the fact above to find the derivative of inverse sine.

$$f(x) = \sin x \qquad g(x) = \sin^{-1} x$$

Then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos\left(\sin^{-1} x\right)}$$

Putting all of this together gives the following derivative.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

b- **Inverse Cosine.**

The inverse cosine and cosine functions are also inverses of each other and so we have,

$$\cos\left(\cos^{-1} x\right) = x \qquad \cos^{-1}\left(\cos x\right) = x$$

To find the derivative we'll do the same kind of work that we did with the inverse sine above. If we start with

$$f(x) = \cos x \qquad g(x) = \cos^{-1} x$$

then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{-\sin\left(\cos^{-1} x\right)}$$

Simplifying the denominator here is almost identical to the work we did for the inverse sine and so isn't shown here. Upon simplifying we get the following derivative.

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

c- ***Inverse Tangent.***

The tangent and inverse tangent functions are inverse functions so,

$$\tan\left(\tan^{-1} x\right) = x \qquad\qquad \tan^{-1}\left(\tan x\right) = x$$

Therefore to find the derivative of the inverse tangent function we can start with

$$f\left(x\right) = \tan x \qquad\qquad g\left(x\right) = \tan^{-1} x$$

We then have,

$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{\sec^2\left(\tan^{-1} x\right)}$$

Simplifying the denominator is similar to the inverse sine, but different enough to warrant showing the details. We'll start with the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad \Rightarrow \qquad \tan y = x$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y$$

Now, if we start with the fact that

$$\cos^2 y + \sin^2 y = 1$$

and divide every term by $\cos^2 y$ we will get,

$$1 + \tan^2 y = \sec^2 y$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y = 1 + \tan^2 y$$

Finally using the second portion of the definition of the inverse tangent function gives us,

$$\sec^2\left(\tan^{-1} x\right) = 1 + \tan^2 y = 1 + x^2$$

The derivative of the inverse tangent is then,

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$$

Here are the derivatives of all six inverse trig functions.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1 + x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{\left|x\right|\sqrt{x^2 - 1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{\left|x\right|\sqrt{x^2 - 1}}$$

Example 2

Differentiate the following functions.

(a) $f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$

(b) $y = \sqrt{z}\sin^{-1}(z)$

Solution

(a) Not much to do with this one other than differentiate each term.

$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

(b) Don't forget to convert the radical to fractional exponents before using the product rule.

$$y' = \frac{1}{2}z^{-\frac{1}{2}}\sin^{-1}(z) + \frac{\sqrt{z}}{\sqrt{1-z^2}}$$

8- ***Derivatives of Hyperbolic Functions***.

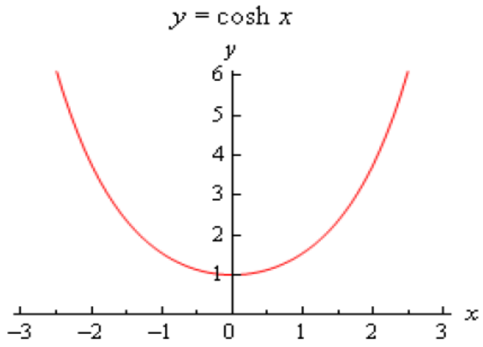
There are six hyperbolic functions and they are defined as follows.

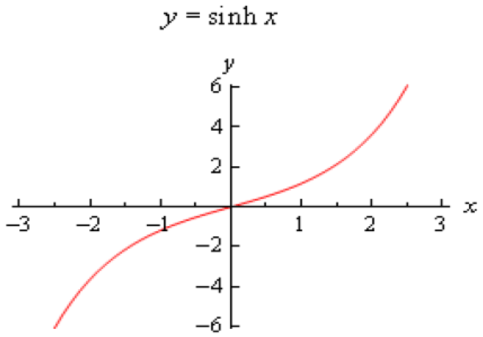
$$\sinh x = \frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2} \qquad \cosh x = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2}$$

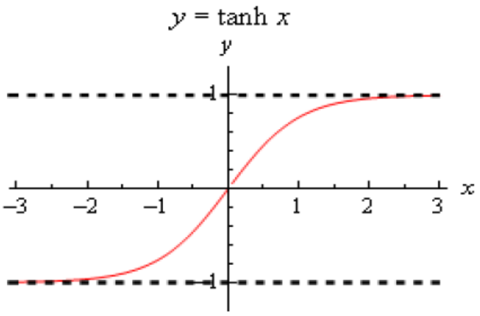
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Here are the graphs of the three main hyperbolic functions.







Because the hyperbolic functions are defined in terms of exponential functions finding their derivatives is fairly simple provided you've already read through the next section. We haven't however so we'll need the following formula that can be easily proved after we've covered the next section.

$$\frac{d}{dx}\left(\mathbf{e}^{-x}\right) = -\mathbf{e}^{-x}$$

With this formula we'll do the derivative for hyperbolic sine and leave the rest to you as an exercise.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2}\right) = \frac{\mathbf{e}^{x} - \left(-\mathbf{e}^{-x}\right)}{2} = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2} = \cosh x$$

For the rest we can either use the definition of the hyperbolic function and/or the quotient rule. Here are all six derivatives.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Example 1 Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \dfrac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$

9- ***<u>Chain Rule.</u>***

Suppose that we have two functions *f(x)* and *g(x)* and they are both differentiable.

1. If we define $F(x) = (f \circ g)(x)$ then the derivative of *F(x)* is,

$$F'(x) = f'(g(x))\; g'(x)$$

2. If we have $y = f(u)$ and $u = g(x)$ then the derivative of *y* is,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of} \\ \text{outside function}}} \underbrace{(g(x))}_{\substack{\text{inside function} \\ \text{left alone}}} \underbrace{g'(x)}_{\substack{\text{times derivative} \\ \text{of inside function}}}$$

Example 1 Use the Chain Rule to differentiate $R(z) = \sqrt{5z-8}$.

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z))\, g'(z) \\ &= f'(5z-8)\, g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}(5) \\ &= \frac{1}{2\sqrt{5z-8}}(5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

Example 2 Differentiate each of the following.

(a) $f(x) = \sin(3x^2 + x)$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$

(d) $g(x) = \ln(x^{-4} + x^4)$

(e) $y = \sec(1 - 5x)$

(f) $P(t) = \cos^4(t) + \cos(t^4)$

Solution

(a) $f(x) = \sin(3x^2 + x)$

It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative of} \\ \text{outside function}}} \underbrace{(3x^2 + x)}_{\substack{\text{leave inside} \\ \text{function alone}}} \underbrace{(6x+1)}_{\substack{\text{times derivative} \\ \text{of inside function}}}$$

Or with a little rewriting,

$$f'(x) = (6x+1)\cos\left(3x^2+x\right)$$

(b) $f(t) = \left(2t^3+\cos(t)\right)^{50}$

In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\begin{aligned} f'(t) &= 50\left(2t^3+\cos(t)\right)^{49}\left(6t^2-\sin(t)\right) \\ &= 50\left(6t^2-\sin(t)\right)\left(2t^3+\cos(t)\right)^{49} \end{aligned}$$

(c) $h(w) = \mathbf{e}^{w^4-3w^2+9}$

$$\begin{aligned} h'(w) &= \mathbf{e}^{w^4-3w^2+9}\left(4w^3-6w\right) \\ &= \left(4w^3-6w\right)\mathbf{e}^{w^4-3w^2+9} \end{aligned}$$

(d) $g(x) = \ln\left(x^{-4}+x^4\right)$

$$g'(x) = \frac{1}{x^{-4}+x^4}\left(-4x^{-5}+4x^3\right) = \frac{-4x^{-5}+4x^3}{x^{-4}+x^4}$$

(e) $y = \sec(1-5x)$

$$\begin{aligned} y' &= \sec(1-5x)\tan(1-5x)(-5) \\ &= -5\sec(1-5x)\tan(1-5x) \end{aligned}$$

(f) $P(t) = \cos^4(t) + \cos\left(t^4\right)$

$$\begin{aligned} P'(t) &= 4\cos^3(t)\left(-\sin(t)\right) - \sin\left(t^4\right)\left(4t^3\right) \\ &= -4\sin(t)\cos^3(t) - 4t^3\sin\left(t^4\right) \end{aligned}$$

Example 3

Differentiate each of the following.

(a) $T(x) = \tan^{-1}(2x)\sqrt[3]{1-3x^2}$

(b) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

Solution

(a) $T(x) = \tan^{-1}(2x)\sqrt[3]{1-3x^2}$

This requires the product rule and each derivative in the product rule will require a chain rule application as well.

$$T'(x) = \frac{1}{1+(2x)^2}(2)\left(1-3x^2\right)^{\frac{1}{3}} + \tan^{-1}(2x)\left(\frac{1}{3}\right)\left(1-3x^2\right)^{-\frac{2}{3}}(-6x)$$

$$= \frac{2\left(1-3x^2\right)^{\frac{1}{3}}}{1+(2x)^2} - 2x\left(1-3x^2\right)^{-\frac{2}{3}}\tan^{-1}(2x)$$

(b) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

In this case we will be using the chain rule in concert with the quotient rule.

$$y' = \frac{5\left(x^3+4\right)^4\left(3x^2\right)\left(1-2x^2\right)^3 - \left(x^3+4\right)^5(3)\left(1-2x^2\right)^2(-4x)}{\left(\left(1-2x^2\right)^3\right)^2}$$

These tend to be a little messy. Notice that when we go to simplify that we'll be able to a fair amount of factoring in the numerator and this will often greatly simplify the derivative.

$$y' = \frac{\left(x^3+4\right)^4\left(1-2x^2\right)^2\left(5\left(3x^2\right)\left(1-2x^2\right) - \left(x^3+4\right)(3)(-4x)\right)}{\left(1-2x^2\right)^6}$$

$$= \frac{3x\left(x^3+4\right)^4\left(5x-6x^3+16\right)}{\left(1-2x^2\right)^4}$$

10- ***Implicit Differentiation***.

They have all been derivatives of functions of the form *y =f (x).*

Example 1 Find y' for $xy = 1$.

<u>*Solution 1 :*</u>

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

<u>*Solution 2 :*</u>

$$xy = x\,y(x) = 1$$

$$\frac{d}{dx}\left(x\,y(x)\right) = \frac{d}{dx}(1)$$

$$(1)\,y(x) + x\frac{d}{dx}\left(y(x)\right) = 0$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}\left(y(x)\right) = \frac{dy}{dx} = y'$$

Using this we get the following,

$$y + xy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = -\frac{1/x}{x} = -\frac{1}{x^2}$$

The process that we used in the second solution to the previous example is called **implicit differentiation** and that is the subject of this section.

Example 2 Differentiate each of the following.

(a) $\left(5x^3 - 7x + 1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

(b) $\sin(3 - 6x)$, $\sin\left(y(x)\right)$

(c) $\mathbf{e}^{x^2-9x}, \quad \mathbf{e}^{y(x)}$

Solution

(a) $(5x^3-7x+1)^5, \ [f(x)]^5, \ [y(x)]^5$

With the first function here we're being asked to do the following,

$$\frac{d}{dx}\left[(5x^3-7x+1)^5\right]=5(5x^3-7x+1)^4(15x^2-7)$$

$$\frac{d}{dx}[f(x)]^5=5[f(x)]^4 f'(x)$$

(b) $\sin(3-6x), \quad \sin(y(x))$

$$\frac{d}{dx}[\sin(3-6x)]=-6\cos(3-6x)$$

$$\frac{d}{dx}[\sin(y(x))]=y'(x)\cos(y(x))$$

(c) $\mathbf{e}^{x^2-9x}, \quad \mathbf{e}^{y(x)}$

$$\frac{d}{dx}\left(\mathbf{e}^{x^2-9x}\right)=(2x-9)\mathbf{e}^{x^2-9x}$$

$$\frac{d}{dx}\left(\mathbf{e}^{y(x)}\right)=y'(x)\mathbf{e}^{y(x)}$$

Example 3 Find y' for the following function.

$$x^2+y^2=9$$

$$\frac{d}{dx}\left(x^2+[y(x)]^2\right)=\frac{d}{dx}(9)$$

All we need to do for the second term is use the chain rule. After taking the derivative we have,

$$2x+2[y(x)]^1 y'(x)=0$$

The final step is to simply solve the resulting equation for y'.

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Example 4 Find the equation of the tangent line to

$$x^2 + y^2 = 9$$

at the point $\left(2, \sqrt{5}\right)$.

The tangent line then is given by,

$$y = f(a) + f'(a)(x - a)$$

Solution

$$m = y'\big|_{x=2,\, y=\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

The tangent line is then.

$$y = \sqrt{5} - \frac{2}{\sqrt{5}}(x - 2)$$

Example 5 Find y' for each of the following.

(a) $x^3y^5 + 3x = 8y^3 + 1$

(b) $x^2 \tan(y) + y^{10} \sec(x) = 2x$

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln\left(xy^3\right)$

Solution

(a) $x^3y^5 + 3x = 8y^3 + 1$

Here is the differentiation of each side for this function.

$$3x^2y^5 + 5x^3y^4y' + 3 = 24y^2y'$$

Then factor y' out of all the terms containing it and divide both sides by the "coefficient" of the y'. Here is the solving work for this one,

$$\begin{aligned} 3x^2y^5 + 3 &= 24y^2y' - 5x^3y^4y' \\ 3x^2y^5 + 3 &= \left(24y^2 - 5x^3y^4\right)y' \\ y' &= \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4} \end{aligned}$$

(b) $x^2\tan\left(y\right) + y^{10}\sec\left(x\right) = 2x$

We've got two product rules to deal with this time. Here is the derivative of this function.

$$2x\tan\left(y\right) + x^2\sec^2\left(y\right)y' + 10y^9y'\sec\left(x\right) + y^{10}\sec\left(x\right)\tan\left(x\right) = 2$$

Now, solve for the derivative.

$$\begin{aligned} \left(x^2\sec^2\left(y\right) + 10y^9\sec\left(x\right)\right)y' &= 2 - y^{10}\sec\left(x\right)\tan\left(x\right) - 2x\tan\left(y\right) \\ y' &= \frac{2 - y^{10}\sec\left(x\right)\tan\left(x\right) - 2x\tan\left(y\right)}{x^2\sec^2\left(y\right) + 10y^9\sec\left(x\right)} \end{aligned}$$

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln\left(xy^3\right)$

Here is the derivative of this equation.

$$\mathbf{e}^{2x+3y}\left(2 + 3y'\right) = 2x - \frac{y^3 + 3xy^2y'}{xy^3}$$

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$\begin{aligned} 2\mathbf{e}^{2x+3y} + 3y'\mathbf{e}^{2x+3y} &= 2x - \frac{y^3}{xy^3} - \frac{3xy^2y'}{xy^3} \\ 2\mathbf{e}^{2x+3y} + 3y'\mathbf{e}^{2x+3y} &= 2x - \frac{1}{x} - \frac{3y'}{y} \\ \left(3\mathbf{e}^{2x+3y} + 3y^{-1}\right)y' &= 2x - x^{-1} - 2\mathbf{e}^{2x+3y} \\ y' &= \frac{2x - x^{-1} - 2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y} + 3y^{-1}} \end{aligned}$$

In the new example we want to look at we're assuming that $x = x\left(t\right)$ and that $y = y\left(t\right)$ and differentiating with respect to t. This means that every time we are faced with an x or a y we'll be doing the chain rule. This in turn means that when we differentiate an x we will need to add on an x' and whenever we differentiate a y we will add on a y'.

Example 6 Assume that $x = x(t)$ and $y = y(t)$ and differentiate the following equation with respect to *t*.

$$x^3y^6 + \mathbf{e}^{1-x} - \cos(5y) = y^2$$

Solution

$$3x^2x'y^6 + 6x^3y^5y' - x'\mathbf{e}^{1-x} + 5y'\sin(5y) = 2yy'$$

11- ***Related Rates.***

In this section we are going to look at an application of implicit differentiation.

Example 1 Air is being pumped into a spherical balloon at a rate of 5 cm³/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution.

Before we do that let's notice that both the volume of the balloon and the radius of the balloon will vary with time and so are really functions of time, *i.e.* $V(t)$ and $r(t)$.

We want to determine the rate at which the radius is changing. Again, rates are derivatives and so it looks like we want to determine,

$$r'(t) = ? \qquad \text{when} \quad r(t) = \frac{d}{2} = 10\,\text{cm}$$

We know that air is being pumped into the balloon at a rate of 5 cm³/min. This is the rate at which the volume is increasing. Recall that rates of change are nothing more than derivatives and so we know that,

$$V'(t) = 5$$

Now that we've identified what we have been given and what we want to find we need to relate these two quantities to each other. In this case we can relate the volume and the radius with the formula for the volume of a sphere.

$$V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$$

$$V' = 4\pi r^2 r'$$

$$5 = 4\pi\left(10^2\right)r' \qquad \Rightarrow \qquad r' = \frac{1}{80\pi} \ \text{cm/min}$$

Let's work another problem that uses some different ideas and shows some of the different kinds of things that can show up in related rates problems.

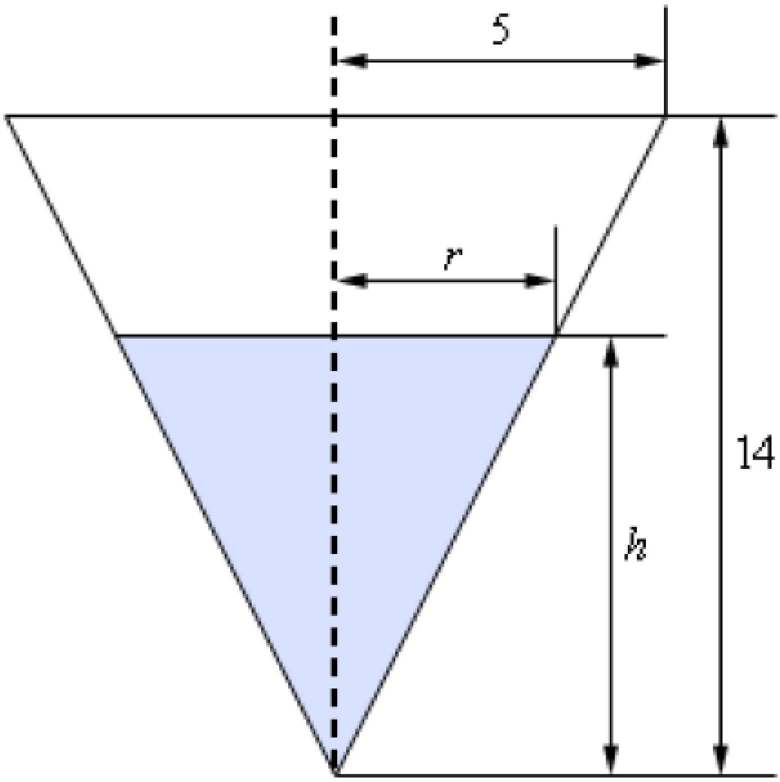
Example-2

A tank of water in the shape of a cone is leaking water at a constant rate of $2\,\text{ft}^3/\text{hour}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

Solution.



The volume of water in the tank at any time t is given by,

$$V = \frac{1}{3}\pi r^2 h$$

and we've been given that $V' = -2$.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

For this part we need to determine h' when $h = 6$ and now we have a problem.

$$V' = \frac{2}{3}\pi r r' h + \frac{1}{3}\pi r^2 h'$$

When we have two similar triangles then ratios of any two sides will be equal. For our set this means that we have,

$$\frac{r}{h} = \frac{5}{14} \qquad \Rightarrow \qquad r = \frac{5}{14}h$$

If we take this and plug it into our volume formula we have,

$$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{5}{14}h\right)^2h=\frac{25}{588}\pi h^3$$

$$V'=\frac{25}{196}\pi h^2h'$$

At this point all we need to do is plug in what we know and solve for h'.

$$-2=\frac{25}{196}\pi\left(6^2\right)h' \qquad\Rightarrow\qquad h'=\frac{-98}{225\pi}=-0.1386$$

So, it looks like the height is decreasing at a rate of 0.1386 ft/hr.

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

In this case we are asking for r'

$$\frac{h}{r}=\frac{14}{5} \qquad\Rightarrow\qquad h=\frac{14}{5}r$$

$$r=\frac{5}{14}h \qquad\Rightarrow\qquad r'=\frac{5}{14}h'$$

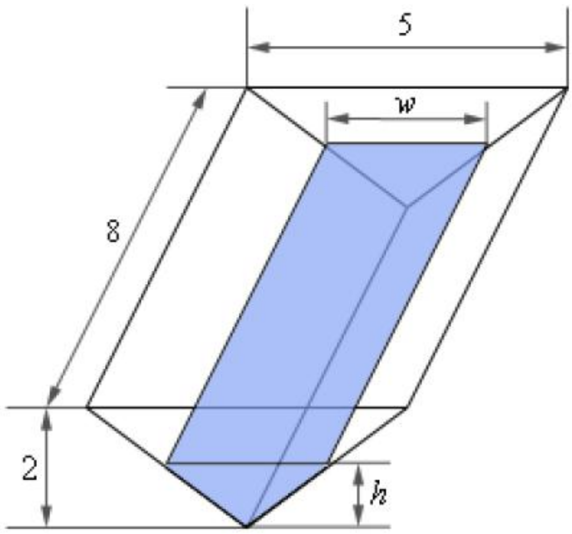
At this point all we need to do here is use the result from the first part to get,

$$r'=\frac{5}{14}\left(\frac{-98}{225\pi}\right)=-\frac{7}{45\pi}=-0.04951$$

Example 3

A trough of water is 8 meters deep and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of $6\ \text{m}^3/\text{sec}$. At what rate is the height of the water changing when the water has a height of 120 cm?

Solution



For our case the volume of the water in the tank is,

$$\begin{aligned} V &= (\text{Area of End})(\text{depth}) \\ &= \left(\tfrac{1}{2}\text{base} \times \text{height}\right)(\text{depth}) \\ &= \tfrac{1}{2}hw(8) \\ &= 4hw \end{aligned}$$

with similar triangles ratios of sides must be equal. In our case we'll use,

$$\frac{w}{5} = \frac{h}{2} \qquad \Rightarrow \qquad w = \frac{5}{2}h$$

Plugging this into the volume gives a formula for the volume (and only for this tank) that only involved the height of the water.

$$V = 4hw = 4h\left(\frac{5}{2}h\right) = 10h^2$$

We can now differentiate this to get,

$$V' = 20hh'$$

Finally, all we need to do is plug in and solve for h'.

$$6 = 20(1.2)h' \qquad \Rightarrow \qquad h' = 0.25 \text{ m/sec}$$

So, the height of the water is rising at a rate of 0.25 m/sec.

12- Higher Order Derivatives.

Let's start this section with the following function.

$$f(x) = 5x^3 - 3x^2 + 10x - 5$$

By this point we should be able to differentiate this function without any problems. Doing this we get,

$$f'(x) = 15x^2 - 6x + 10$$

Now, this is a function and so it can be differentiated. Here is the notation that we'll use for that, as well as the derivative.

$$f''(x) = \left(f'(x)\right)' = 30x - 6$$

This is called the **second derivative** and $f'(x)$ is now called the **first derivative**.

Again, this is a function so we can differentiate it again. This will be called the **third derivative**. Here is that derivative as well as the notation for the third derivative.

$$f'''(x) = \left(f''(x)\right)' = 30$$

Collectively the second, third, fourth, *etc*. derivatives are called **higher order derivatives**.

Example 1 Find the first four derivatives for each of the following.

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

(b) $y = \cos x$

(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$

Solution

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

There really isn't a lot to do here other than do the derivatives.

$$R'(t) = 6t + 4t^{-\frac{1}{2}} + \mathbf{e}^t$$

$$R''(t) = 6 - 2t^{-\frac{3}{2}} + \mathbf{e}^t$$

$$R'''(t) = 3t^{-\frac{5}{2}} + \mathbf{e}^t$$

$$R^{(4)}(t) = -\frac{15}{2}t^{-\frac{7}{2}} + \mathbf{e}^t$$

(b) $y = \cos x$

Again, let's just do some derivatives.

$$\begin{aligned} y &= \cos x \\ y' &= -\sin x \\ y'' &= -\cos x \\ y''' &= \sin x \\ y^{(4)} &= \cos x \end{aligned}$$

(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$

$$f'(y) = 3\cos(3y) - 2\mathbf{e}^{-2y} + \frac{1}{y} = 3\cos(3y) - 2\mathbf{e}^{-2y} + y^{-1}$$

$$f''(y) = -9\sin(3y) + 4\mathbf{e}^{-2y} - y^{-2}$$

$$f'''(y) = -27\cos(3y) - 8\mathbf{e}^{-2y} + 2y^{-3}$$

$$f^{(4)}(y) = 81\sin(3y) + 16\mathbf{e}^{-2y} - 6y^{-4}$$

Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called **logarithmic differentiation**.

Example 1 Differentiate the function.

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. However, that would be a fairly messy process. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$

$$\ln y = \ln\left(x^5\right) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

What we need to do at this point is differentiate both sides with respect to *x*. Note that this is really implicit differentiation.

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+2\right)^{-\frac{1}{2}}(2x)}{\left(x^2+2\right)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

To finish the problem all that we need to do is multiply both sides by y and the plug in for y since we do know what that is.

$$\begin{aligned} y' &= y\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right) \\ &= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right) \end{aligned}$$

Example 3 Differentiate $y = (1-3x)^{\cos(x)}$

Solution

Now, this looks much more complicated than the previous example, but is in fact only slightly more complicated. The process is pretty much identical so we first take the log of both sides and then simplify the right side.

$$\ln y = \ln\left[(1-3x)^{\cos(x)}\right] = \cos(x)\ln(1-3x)$$

Next, do some implicit differentiation.

$$\frac{y'}{y} = -\sin(x)\ln(1-3x) + \cos(x)\frac{-3}{1-3x} = -\sin(x)\ln(1-3x) - \cos(x)\frac{3}{1-3x}$$

Finally, solve for y' and substitute back in for y.

$$\begin{aligned} y' &= -y\left(\sin(x)\ln(1-3x) + \cos(x)\frac{3}{1-3x}\right) \\ &= -(1-3x)^{\cos(x)}\left(\sin(x)\ln(1-3x) + \cos(x)\frac{3}{1-3x}\right) \end{aligned}$$

Problems: ***Sheet No. 4***

13- Problems.

For problems 1 – 12 find the derivative of the given function.

1. $f(x) = 6x^3 - 9x + 4$

2. $y = 2t^4 - 10t^2 + 13t$

3. $g(z) = 4z^7 - 3z^{-7} + 9z$

4. $h(y) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$

5. $y = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$

6. $f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$

7. $f(t) = \dfrac{4}{t} - \dfrac{1}{6t^3} + \dfrac{8}{t^5}$

8. $R(z) = \dfrac{6}{\sqrt{z^3}} + \dfrac{1}{8z^4} - \dfrac{1}{3z^{10}}$

9. $z = x(3x^2 - 9)$

10. $g(y) = (y-4)(2y+y^2)$

11. $h(x) = \dfrac{4x^3 - 7x + 8}{x}$

12. $f(y) = \dfrac{y^5 - 5y^3 + 2y}{y^3}$

13

Find the tangent line to $f(x) = 7x^4 + 8x^{-6} + 2x$ at $x = -1$.

Problems: ***Sheet No. 4***

14 -

The position of an object at any time t is given by $s(t) = 3t^4 - 40t^3 + 126t^2 - 9$.

(a) Determine the velocity of the object at any time t.

(b) Does the object ever stop changing?

(c) When is the object moving to the right and when is the object moving to the left?

Product and Quotient Rule

For problems 1 – 6 use the Product Rule or the Quotient Rule to find the derivative of the given function.

1. $f(t) = (4t^2 - t)(t^3 - 8t^2 + 12)$

2. $y = \left(1 + \sqrt{x^3}\right)\left(x^{-3} - 2\sqrt[3]{x}\right)$

3. $h(z) = (1 + 2z + 3z^2)(5z + 8z^2 - z^3)$

4. $g(x) = \dfrac{6x^2}{2 - x}$

5. $R(w) = \dfrac{3w + w^4}{2w^2 + 1}$

6. $f(x) = \dfrac{\sqrt{x} + 2x}{7x - 4x^2}$

7. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$ determine the value of $(f\,g)'(2)$.

8. If $f(x) = x^3 g(x)$, $g(-7) = 2$, $g'(-7) = -9$ determine the value of $f'(-7)$.

9. Find the equation of the tangent line to $f(x) = \left(1 + 12\sqrt{x}\right)(4 - x^2)$ at $x = 9$.

10. Determine where $f(x) = \dfrac{x - x^2}{1 + 8x^2}$ is increasing and decreasing.

11. Determine where $V(t) = (4 - t^2)(1 + 5t^2)$ is increasing and decreasing.

Problems: ***Sheet No. 4***

Derivatives of Trig Functions

For problems 1 – 3 evaluate the given limit.

1. $\lim\limits_{z \to 0} \dfrac{\sin(10z)}{z}$

2. $\lim\limits_{\alpha \to 0} \dfrac{\sin(12\alpha)}{\sin(5\alpha)}$

3. $\lim\limits_{x \to 0} \dfrac{\cos(4x)-1}{x}$

For problems 4 – 10 differentiate the given function.

4. $f(x) = 2\cos(x) - 6\sec(x) + 3$

5. $g(z) = 10\tan(z) - 2\cot(z)$

6. $f(w) = \tan(w)\sec(w)$

7. $h(t) = t^3 - t^2\sin(t)$

8. $y = 6 + 4\sqrt{x}\csc(x)$

9. $R(t) = \dfrac{1}{2\sin(t) - 4\cos(t)}$

10. $Z(v) = \dfrac{v + \tan(v)}{1 + \csc(v)}$

Derivatives of Exponential and Logarithm Functions

For problems 1 – 6 differentiate the given function.

1. $f(x) = 2\mathbf{e}^x - 8^x$

2. $g(t) = 4\log_3(t) - \ln(t)$

3. $R(w) = 3^w \log(w)$

4. $y = z^5 - \mathbf{e}^z \ln(z)$

Problems: **_Sheet No. 4_**

5. $h(y) = \dfrac{y}{1 - \mathbf{e}^y}$

6. $f(t) = \dfrac{1 + 5t}{\ln(t)}$

7. Find the tangent line to $f(x) = 7^x + 4\mathbf{e}^x$ at $x = 0$.

8. Find the tangent line to $f(x) = \ln(x)\log_2(x)$ at $x = 2$.

9. Determine if $V(t) = \dfrac{t}{\mathbf{e}^t}$ is increasing or decreasing at the following points.

(a) $t = -4$ **(b)** $t = 0$ **(c)** $t = 10$

Derivatives of Inverse Trig Functions

For each of the following problems differentiate the given function.

1. $T(z) = 2\cos(z) + 6\cos^{-1}(z)$
2. $g(t) = \csc^{-1}(t) - 4\cot^{-1}(t)$
3. $y = 5x^6 - \sec^{-1}(x)$
4. $f(w) = \sin(w) + w^2 \tan^{-1}(w)$
5. $h(x) = \dfrac{\sin^{-1}(x)}{1 + x}$

Derivatives of Hyperbolic Functions

For each of the following problems differentiate the given function.

1. $f(x) = \sinh(x) + 2\cosh(x) - \operatorname{sech}(x)$
2. $R(t) = \tan(t) + t^2 \operatorname{csch}(t)$
3. $g(z) = \dfrac{z + 1}{\tanh(z)}$

Problems: ***Sheet No. 4***

Chain Rule

differentiate the given function.

1. $f(x) = (6x^2 + 7x)^4$

2. $g(t) = (4t^2 - 3t + 2)^{-2}$

3. $y = \sqrt[3]{1 - 8z}$

4. $R(w) = \csc(7w)$

5. $G(x) = 2\sin(3x + \tan(x))$

6. $h(u) = \tan(4 + 10u)$

7. $f(t) = 5 + \mathbf{e}^{4t + t^7}$

8. $g(x) = \mathbf{e}^{1 - \cos(x)}$

9. $H(z) = 2^{1 - 6z}$

10. $u(t) = \tan^{-1}(3t - 1)$

11. $F(y) = \ln(1 - 5y^2 + y^3)$

12. $V(x) = \ln(\sin(x) - \cot(x))$

13. $h(z) = \sin(z^6) + \sin^6(z)$

Problems: ***Sheet No. 4***

14. $S(w) = \sqrt{7w} + \mathbf{e}^{-w}$

15. $g(z) = 3z^7 - \sin(z^2 + 6)$

16. $f(x) = \ln(\sin(x)) - (x^4 - 3x)^{10}$

17. $h(t) = t^6 \sqrt{5t^2 - t}$

18. $q(t) = t^2 \ln(t^5)$

19. $g(w) = \cos(3w)\sec(1 - w)$

20. $y = \dfrac{\sin(3t)}{1 + t^2}$

21. $K(x) = \dfrac{1 + \mathbf{e}^{-2x}}{x + \tan(12x)}$

22. $f(x) = \cos(x^2 \mathbf{e}^x)$

23. $z = \sqrt{5x + \tan(4x)}$

24. $f(t) = \left(\mathbf{e}^{-6t} + \sin(2 - t)\right)^3$

25. $g(x) = \left(\ln(x^2 + 1) - \tan^{-1}(6x)\right)^{10}$

26. $h(z) = \tan^4(z^2 + 1)$

27. $f(x) = \left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-1}$

28. Find the tangent line to $f(x) = 4\sqrt{2x} - 6\mathbf{e}^{2-x}$ at $x = 2$.

29. Determine where $V(z) = z^4(2z - 8)^3$ is increasing and decreasing.

Problems: ***Sheet No. 4***

Implicit Differentiation

For problems 1 – 3 do each of the following.

(a) Find y' by solving the equation for y and differentiating directly.

(b) Find y' by implicit differentiation.

(c) Check that the derivatives in **(a)** and **(b)** are the same.

1. $\frac{x}{y^3} = 1$

2. $x^2 + y^3 = 4$

3. $x^2 + y^2 = 2$

For problems 4 – 9 find y' by implicit differentiation.

4. $2y^3 + 4x^2 - y = x^6$

5. $7y^2 + \sin(3x) = 12 - y^4$

6. $\mathbf{e}^x - \sin(y) = x$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

8. $\cos(x^2 + 2y) + x\,\mathbf{e}^{y^2} = 1$

9. $\tan(x^2y^4) = 3x + y^2$

Problems: ***Sheet No. 4***

2. $x^2 + y^3 = 4$

3. $x^2 + y^2 = 2$

For problems 4 – 9 find y' by implicit differentiation.

4. $2y^3 + 4x^2 - y = x^6$

5. $7y^2 + \sin(3x) = 12 - y^4$

6. $\mathbf{e}^x - \sin(y) = x$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

8. $\cos(x^2 + 2y) + x\,\mathbf{e}^{y^2} = 1$

9. $\tan(x^2y^4) = 3x + y^2$

For problems 10 & 11 find the equation of the tangent line at the given point.

10. $x^4 + y^2 = 3$ at $\left(1, -\sqrt{2}\right)$.

11. $y^2\mathbf{e}^{2x} = 3y + x^2$ at $(0,3)$.

For problems 12 & 13 assume that $x = x(t)$, $y = y(t)$ and $z = z(t)$ and differentiate the given equation with respect to *t*.

12. $x^2 - y^3 + z^4 = 1$

13. $x^2\cos(y) = \sin(y^3 + 4z)$

Related Rates

1. In the following assume that *x* and *y* are both functions of *t*. Given $x = -2$, $y = 1$ and $x' = -4$ determine y' for the following equation.

Problems: ***Sheet No. 4***

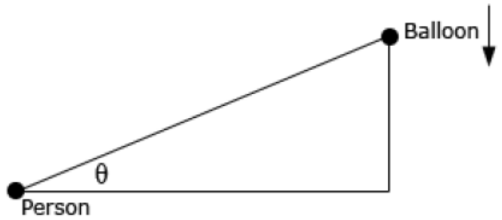
$$6y^2 + x^2 = 2 - x^3\mathbf{e}^{4-4y}$$

2. In the following assume that x, y and z are all functions of t. Given $x = 4$, $y = -2$, $z = 1$, $x' = 9$ and $y' = -3$ determine z' for the following equation.

$$x(1-y) + 5z^3 = y^2z^2 + x^2 - 3$$

10. A tank of water in the shape of a cone is being filled with water at a rate of 12 m³/sec. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing with the radius of the top of the water is 10 meters?

11. The angle of elevation is the angle formed by a horizontal line and a line joining the observer's eye to an object above the horizontal line. A person is 500 feet way from the launch point of a hot air balloon. The hot air balloon is starting to come back down at a rate of 15 ft/sec. At what rate is the angle of elevation, θ, changing when the hot air balloon is 200 feet above the ground. See the (probably bad) sketch below to help visualize the angle of elevation if you are having trouble seeing it.



Higher Order Derivatives

For problems 1 – 5 determine the fourth derivative of the given function.

1. $h(t) = 3t^7 - 6t^4 + 8t^3 - 12t + 18$

2. $V(x) = x^3 - x^2 + x - 1$

3. $f(x) = 4\sqrt[5]{x^3} - \frac{1}{8x^2} - \sqrt{x}$

4. $f(w) = 7\sin\left(\frac{w}{3}\right) + \cos(1-2w)$

5. $y = \mathbf{e}^{-5z} + 8\ln\left(2z^4\right)$

For problems 6 – 9 determine the second derivative of the given function.

Problems: ***Sheet No. 4***

6. $g(x) = \sin(2x^3 - 9x)$

7. $z = \ln(7 - x^3)$

8. $Q(v) = \dfrac{2}{(6 + 2v - v^2)^4}$

9. $H(t) = \cos^2(7t)$

Logarithmic Differentiation

For problems 1 – 3 use logarithmic differentiation to find the first derivative of the given function.

1. $f(x) = (5 - 3x^2)^7 \sqrt{6x^2 + 8x - 12}$

2. $y = \dfrac{\sin(3z + z^2)}{(6 - z^4)^3}$

3. $h(t) = \dfrac{\sqrt{5t + 8}\ \sqrt[3]{1 - 9\cos(4t)}}{\sqrt[4]{t^2 + 10t}}$

For problems 4 & 5 find the first derivative of the given function.

4. $g(w) = (3w - 7)^{4w}$

5. $f(x) = (2x - \mathbf{e}^{8x})^{\sin(2x)}$