CALCULAS-II

First year/ 2nd Semester - Chemical and Petroleum Engineering Department

By

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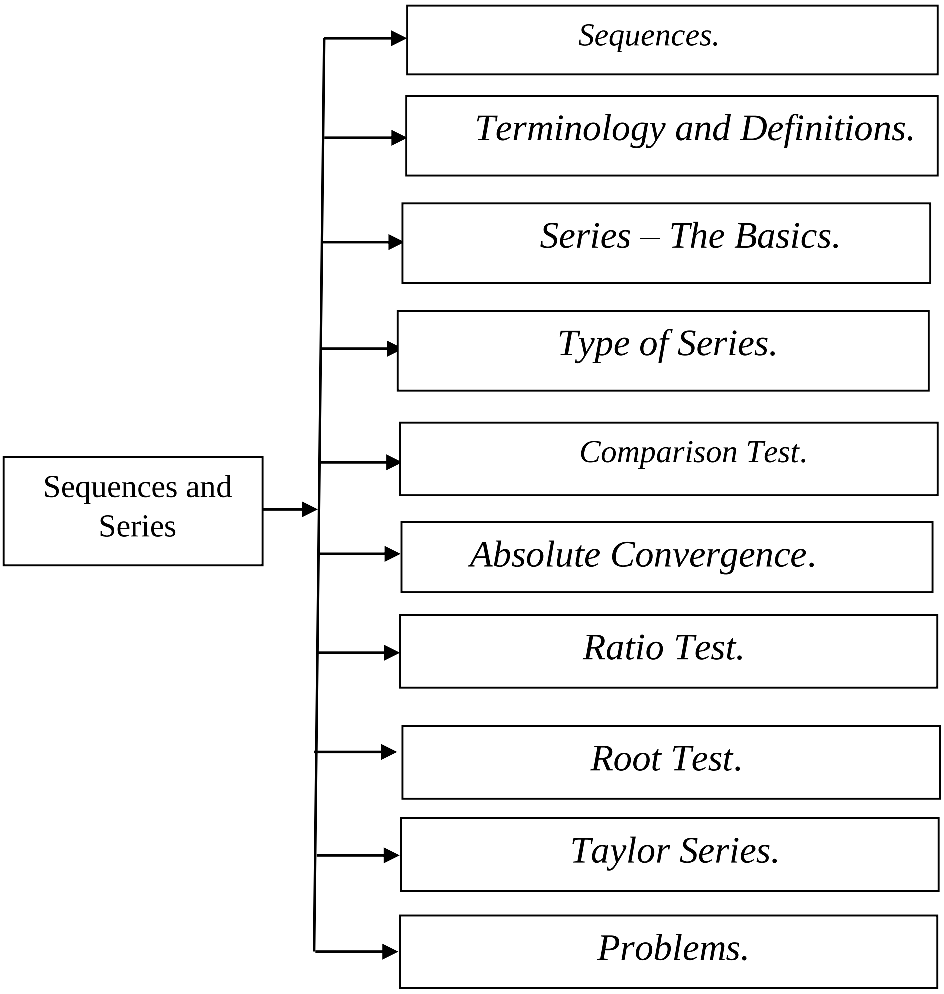
Lecture – Five

***Sequences* and Series**



1- Sequences.

A sequence is nothing more than a list of numbers written in a specific order. General sequence terms are denoted as follows,

$$\begin{aligned}
&a_1 - \text{first term}\\
&a_2 - \text{second term}\\
&\quad\vdots\\
&a_n - n^{th} \text{ term}\\
&a_{n+1} - (n+1)^{\text{st}} \text{ term}
\end{aligned}$$

There is a variety of ways of denoting a sequence. Each of the following are equivalent ways of denoting a sequence.

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

Example 1 Write down the first few terms of each of the following sequences.

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$ [Solution]

(b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$ [Solution]

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π [Solution]

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots\right\}$$

(b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

This one is similar to the first one. The main difference is that this sequence doesn't start at $n=1$.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right\}$$

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

So we know that $\pi = 3.14159265359\ldots$

The sequence is then,

$$\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots\}$$

Theorem 1

Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$

Theorem 2

If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Theorem 3

The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \le 1$ and diverges for all other value of r. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Example 2 Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

(a) $\left\{ \dfrac{3n^2 - 1}{10n + 5n^2} \right\}_{n=2}^{\infty}$ [Solution]

(b) $\left\{ \dfrac{\mathbf{e}^{2n}}{n} \right\}_{n=1}^{\infty}$ [Solution]

(c) $\left\{ \dfrac{(-1)^n}{n} \right\}_{n=1}^{\infty}$ [Solution]

(d) $\left\{ (-1)^n \right\}_{n=0}^{\infty}$ [Solution]

Solution

(a) $\left\{ \dfrac{3n^2 - 1}{10n + 5n^2} \right\}_{n=2}^{\infty}$

To do a limit in this form all we need to do is factor from the numerator and denominator the largest power of n, cancel and then take the limit.

$$\lim_{n\to\infty} \frac{3n^2 - 1}{10n + 5n^2} = \lim_{n\to\infty} \frac{n^2\left(3 - \frac{1}{n^2}\right)}{n^2\left(\frac{10}{n} + 5\right)} = \lim_{n\to\infty} \frac{3 - \frac{1}{n^2}}{\frac{10}{n} + 5} = \frac{3}{5}$$

So the sequence converges and its limit is $\frac{3}{5}$.

(b) $\left\{ \dfrac{\mathbf{e}^{2n}}{n} \right\}_{n=1}^{\infty}$

Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x) = \frac{\mathbf{e}^{2x}}{x}$$

and note that,

$$f(n) = \frac{\mathbf{e}^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty}\frac{\mathbf{e}^{2n}}{n}=\lim_{x\to\infty}\frac{\mathbf{e}^{2x}}{x}=\lim_{x\to\infty}\frac{2\mathbf{e}^{2x}}{1}=\infty$$

So, the sequence in this part diverges (to ∞).

(c) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

We will need to use Theorem 2 on this problem. To this

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|=\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore, since the limit of the sequence terms with absolute value bars on them goes to zero we know by Theorem 2 that,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=0$$

which also means that the sequence converges to a value of zero.

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

For this theorem note that all we need to do is realize that this is the sequence in Theorem 3 above using $r=-1$. So, by Theorem 3 this sequence diverges.

Theorem 4

For the sequence $\{a_n\}$ if both $\lim_{n\to\infty}a_{2n}=L$ and $\lim_{n\to\infty}a_{2n+1}=L$ then $\{a_n\}$ is convergent and $\lim_{n\to\infty}a_n=L$.

2- *Terminology and Definitions.*

Let's start off with some terminology and definitions.

Given any sequence $\{a_n\}$ we have the following.

1. We call the sequence **increasing** if $a_n < a_{n+1}$ for every n.
2. We call the sequence **decreasing** if $a_n > a_{n+1}$ for every n.
3. If $\{a_n\}$ is an increasing sequence or $\{a_n\}$ is a decreasing sequence we call it **monotonic**.
4. If there exists a number m such that $m \le a_n$ for every n we say the sequence is **bounded below**. The number m is sometimes called a **lower bound** for the sequence.
5. If there exists a number M such that $a_n \le M$ for every n we say the sequence is **bounded above**. The number M is sometimes called an **upper bound** for the sequence.
6. If the sequence is both bounded below and bounded above we call the sequence **bounded**.

Example 1 Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$ [Solution]

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$ [Solution]

(c) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$ [Solution]

Solution

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) because,

$$-n^2 > -(n+1)^2$$

for every n.

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

The sequence terms in this sequence alternate between 1 and -1 and so the sequence is neither an increasing sequence or a decreasing sequence. Since the sequence is neither an increasing nor decreasing sequence it is not a monotonic sequence.

The sequence is bounded however since it is bounded above by 1 and bounded below by -1.

(c) $\left\{ \frac{2}{n^2} \right\}_{n=5}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) since,

$$\frac{2}{n^2} > \frac{2}{(n+1)^2}$$

The terms in this sequence are all positive and so it is bounded below by zero. Also, since the sequence is a decreasing sequence the first sequence term will be the largest and so we can see that the sequence will also be bounded above by $\frac{2}{25}$. Therefore, this sequence is bounded.

We can also take a quick limit and note that this sequence converges and its limit is zero.

Example 2 Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$ [Solution]

(b) $\left\{ \frac{n^3}{n^4+10000} \right\}_{n=0}^{\infty}$ [Solution]

Solution

(a) $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

To determine the increasing/decreasing nature of this sequence we will need to resort to Calculus I techniques. First consider the following function and its derivative.

$$f(x) = \frac{x}{x+1} \qquad\qquad f'(x) = \frac{1}{(x+1)^2}$$

We can see that the first derivative is always positive and so from Calculus I we know that the function must then be an increasing function. So, how does this help us? Notice that,

$$f(n) = \frac{n}{n+1} = a_n$$

Therefore because $n < n+1$ and $f(x)$ is increasing we can also say that,

$$a_n = \frac{n}{n+1} = f(n) < f(n+1) = \frac{n+1}{n+1+1} = a_{n+1} \qquad \Rightarrow \qquad a_n < a_{n+1}$$

In other words, the sequence must be increasing.

Note that now that we know the sequence is an increasing sequence we can get a better lower bound for the sequence. Since the sequence is increasing the first term in the sequence must be the smallest term and so since we are starting at $n = 1$ we could also use a lower bound of $\frac{1}{2}$ for this sequence. It is important to remember that any number that is always less than or equal to all the sequence terms can be a lower bound. Some are better than others however.

A quick limit will also tell us that this sequence converges with a limit of 1.

(b) $\left\{ \dfrac{n^3}{n^4 + 10000} \right\}_{n=0}^{\infty}$

This however, isn't a decreasing sequence. Let's take a look at the first few terms to see this.

$$a_1 = \frac{1}{10001} \approx 0.00009999 \qquad a_2 = \frac{1}{1252} \approx 0.0007987$$

$$a_3 = \frac{27}{10081} \approx 0.005678 \qquad a_4 = \frac{4}{641} \approx 0.006240$$

$$a_5 = \frac{1}{85} \approx 0.011756 \qquad a_6 = \frac{27}{1412} \approx 0.019122$$

$$a_7 = \frac{343}{12401} \approx 0.02766 \qquad a_8 = \frac{32}{881} \approx 0.03632$$

$$a_9 = \frac{729}{16561} \approx 0.04402 \qquad a_{10} = \frac{1}{20} = 0.05$$

Now, we can't make another common mistake and assume that because the first few terms increase then whole sequence must also increase. If we did that we would also be mistaken as this is also not an increasing sequence.

This sequence is neither decreasing or increasing. The only sure way to see this is to do the Calculus I approach to increasing/decreasing functions.

In this case we'll need the following function and its derivative.

$$f(x) = \frac{x^3}{x^4 + 10000} \qquad f'(x) = \frac{-x^2\left(x^4 - 30000\right)}{\left(x^4 + 10000\right)^2}$$

This function will have the following three critical points,

$$x = 0, \quad x = \sqrt[4]{30000} \approx 13.1607, \quad x = -\sqrt[4]{30000} \approx -13.1607$$

Why critical points? Remember these are the only places where the function *may* change sign! Our sequence starts at $n = 0$ and so we can ignore the third one since it lies outside the values of n that we're considering. By plugging in some test values of x we can quickly determine that the derivative is positive for $0 < x < \sqrt[4]{30000} \approx 13.16$ and so the function is increasing in this range. Likewise, we can see that the derivative is negative for $x > \sqrt[4]{30000} \approx 13.16$ and so the function will be decreasing in this range.

So, our sequence will be increasing for $0 \le n \le 13$ and decreasing for $n \ge 13$. Therefore the function is not monotonic.

3- ***Series – The Basics.***

That topic is infinite series. So just what is an infinite series? Well, let's start with a sequence $\{a_n\}_{n=1}^{\infty}$ (note the $n = 1$ is for convenience, it can be anything) and define the following,

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i
\end{aligned}$$

The s_n are called **partial sums** and notice that they will form a sequence, $\{s_n\}_{n=1}^{\infty}$. Also recall that the Σ is used to represent this summation and called a variety of names. The most common names are : **series notation**, **summation notation**, and **sigma notation**.

We want to take a look at the limit of the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$.

Notationally we'll define,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

If the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$, is convergent and its limit is finite then we also call the infinite series, $\sum_{i=1}^{\infty} a_i$ **convergent** and if the sequence of partial sums is divergent then the infinite series is also called **divergent**.

Note that sometimes it is convenient to write the infinite series as,

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

So, we've determined the convergence of four series now. Two of the series converged and two diverged. Let's go back and examine the series terms for each of these. For each of the series let's take the limit as n goes to infinity of the series terms (not the partial sums!!).

$\lim\limits_{n\to\infty} n = \infty$	this series diverged
$\lim\limits_{n\to\infty} \dfrac{1}{n^2-1} = 0$	this series converged
$\lim\limits_{n\to\infty} (-1)^n$ doesn't exist	this series diverged
$\lim\limits_{n\to\infty} \dfrac{1}{3^{n-1}} = 0$	this series converged

Theorem

If $\sum a_n$ converges then $\lim\limits_{n\to\infty} a_n = 0$.

Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge.

Example -1 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

That's what we'll do here.

$$\lim_{n\to\infty} \frac{4n^2 - n^3}{10 + 2n^3} = -\frac{1}{2} \neq 0$$

The limit of the series terms isn't zero and so by the Divergence Test the series diverges.

4- <u>Type of Series.</u>

A- ***<u>Geometric Series.</u>***

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or, with an index shift the geometric series will often be written as,

$$\sum_{n=0}^{\infty} ar^{n}$$

These are identical series and will have identical values, provided they converge of course.

Recall that by multiplying S_n by r and subtracting the result from S_n one obtains

If we start with the first form it can be shown that the partial sums are,

$$s_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

The series will converge provided the partial sums form a convergent sequence, so let's take the limit of the partial sums.

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right)\\ &= \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{ar^n}{1-r}\\ &= \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n\end{aligned}$$

Now, from Theorem 3 from the Sequences section we know that the limit above will exist and be finite provided $-1 < r \le 1$. However, note that we can't let $r = 1$ since this will give division by zero. Therefore, this will exist and be finite provided $-1 < r < 1$ and in this case the limit is zero and so we get,

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

Therefore, a geometric series will converge if $-1 < r < 1$, which is usually written $|r| < 1$, its value is,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$$

Example 1 Determine if the following series converge or diverge. If they converge give the value of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

So, let's first get rid of that.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n-2)}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}}$$

Now let's get the correct exponent on each of the numbers. This can be done using simple exponent properties.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n-1}4^{2}}{9^{n-1}9^{-1}}$$

Now, rewrite the term a little.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 16(9)\frac{4^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$$

So, this is a geometric series with $a = 144$ and $r = \frac{4}{9} < 1$. Therefore, since $|r| < 1$ we know the series will converge and its value will be,

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \frac{144}{1-\frac{4}{9}} = \frac{9}{5}(144) = \frac{1296}{5}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

$$\sum_{n=0}^{\infty}\frac{(-4)^{3n}}{5^{n-1}}=\sum_{n=0}^{\infty}\frac{\left((-4)^3\right)^n}{5^n5^{-1}}=\sum_{n=0}^{\infty}5\frac{(-64)^n}{5^n}=\sum_{n=0}^{\infty}5\left(\frac{-64}{5}\right)^n$$

So, we've got it into the correct form and we can see that $a=5$ and $r=-\frac{64}{5}$. Also note that $|r|\geq 1$ and so this series diverges.

Example 2 Use the results from the previous example to determine the value of the following series.

(a) $\sum_{n=0}^{\infty}9^{-n+2}4^{n+1}$

(b) $\sum_{n=3}^{\infty}9^{-n+2}4^{n+1}$

Solution

(a) $\sum_{n=0}^{\infty}9^{-n+2}4^{n+1}$

Let's notice that if we strip out the first term from this series we arrive at,

$$\sum_{n=0}^{\infty}9^{-n+2}4^{n+1}=9^24^1+\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}=324+\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}$$

From the previous example we know the value of the new series that arises here and so the value of the series in this example is,

$$\sum_{n=0}^{\infty}9^{-n+2}4^{n+1}=324+\frac{1296}{5}=\frac{2916}{5}$$

(b) $\sum_{n=3}^{\infty}9^{-n+2}4^{n+1}$

$$\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}=9^14^2+9^04^3+\sum_{n=3}^{\infty}9^{-n+2}4^{n+1}=208+\sum_{n=3}^{\infty}9^{-n+2}4^{n+1}$$

We can now use the value of the series from the previous example to get the value of this series.

$$\sum_{n=3}^{\infty}9^{-n+2}4^{n+1}=\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}-208=\frac{1296}{5}-208=\frac{256}{5}$$

B- ***Power Series***.

Fact (The *p* –series Test)

If $k>0$ then $\sum_{n=k}^{\infty}\frac{1}{n^p}$ converges if $p>1$ and diverges if $p\le 1$.

Using the *p*-series test makes it very easy to determine the convergence of some series.

Example 3 Determine if the following series are convergent or divergent.

(a) $\sum_{n=4}^{\infty}\frac{1}{n^7}$

(b) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

Solution

(a) In this case $p=7>1$ and so by this fact the series is convergent.

(b) For this series $p=\frac{1}{2}\le 1$ and so the series is divergent by the fact.

In this section we are going to start talking about power series. A **power series about a**, or just **power series**, is any series that can be written in the form,

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

The c_n's are often called the **coefficients** of the series.

First, as we will see in our examples, we will be able to show that there is a number R so that the power series will converge for, $|x-a|<R$ and will diverge for $|x-a|>R$. This number is called the **radius of convergence** for the series. Note that the series may or may not converge if $|x-a|=R$. What happens at these points will not change the radius of convergence.

Secondly, the interval of all x's, including the endpoints if need be, for which the power series converges is called the **interval of convergence** of the series.

These two concepts are fairly closely tied together. If we know that the radius of convergence of a power series is R then we have the following.

$a-R<x<a+R$	power series converges
$x<a-R$ and $x>a+R$	power series diverges

The interval of convergence must then contain the interval $a-R<x<a+R$ since we know that the power series will converge for these values.

Before getting into some examples let's take a quick look at the convergence of a power series for the case of $x = a$. In this case the power series becomes,

$$\sum_{n=0}^{\infty} c_n (a-a)^n = \sum_{n=0}^{\infty} c_n (0)^n = c_0 (0)^0 + \sum_{n=1}^{\infty} c_n (0)^n = c_0 + \sum_{n=1}^{\infty} 0 = c_0 + 0 = c_0$$

and so the power series converges. Note that we had to strip out the first term since it was the only non-zero term in the series.

Example 1 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n$$

Solution

With all that said, the best tests to use here are almost always the ratio or root test. Most of the power series that we'll be looking at are set up for one or the other. In this case we'll use the ratio test.

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (n+1)(x+3)^{n+1}}{4^{n+1}} \frac{4^n}{(-1)^n (n)(x+3)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{-(n+1)(x+3)}{4n} \right|$$

The limit is then,

$$L = |x+3| \lim_{n \to \infty} \frac{n+1}{4n}$$

$$= \frac{1}{4}|x+3|$$

So, the ratio test tells us that if $L < 1$ the series will converge, if $L > 1$ the series will diverge, and if $L = 1$ we don't know what will happen. So, we have,

$$\frac{1}{4}|x+3| < 1 \qquad \Rightarrow \qquad |x+3| < 4 \qquad\qquad \text{series converges}$$

$$\frac{1}{4}|x+3| > 1 \qquad \Rightarrow \qquad |x+3| > 4 \qquad\qquad \text{series diverges}$$

radius of convergence for this power series is $R = 4$.

Now, let's get the interval of convergence. We'll get most (if not all) of the interval by solving the first inequality from above.

$$-4 < x+3 < 4$$

$$-7 < x < 1$$

The way to determine convergence at these points is to simply plug them into the original power series and see if the series converges or diverges using any test necessary.

$x = -7$:
In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (-1)^n 4^n$$
$$= \sum_{n=1}^{\infty} (-1)^n (-1)^n n \qquad (-1)^n (-1)^n = (-1)^{2n} = 1$$
$$= \sum_{n=1}^{\infty} n$$

This series is divergent by the Divergence Test since $\lim_{n\to\infty} n = \infty \neq 0$.

$x = 1$:
In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (4)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This series is also divergent by the Divergence Test since $\lim_{n\to\infty} (-1)^n n$ doesn't exist.

So, in this case the power series will not converge for either endpoint. The interval of convergence is then,

$$-7 < x < 1$$

Example 2 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n} (4x-8)^n$$

Solution
Let's jump right into the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{2^{n+1} (4x-8)^{n+1}}{n+1} \frac{n}{2^n (4x-8)^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{2n(4x-8)}{n+1} \right|$$
$$= |4x-8| \lim_{n\to\infty} \frac{2n}{n+1}$$
$$= 2|4x-8|$$

So we will get the following convergence/divergence information from this.

$$2|4x-8| < 1 \qquad \text{series converges}$$
$$2|4x-8| > 1 \qquad \text{series diverges}$$

We need to be careful here in determining the interval of convergence. The interval of convergence requires $|x-a| < R$ and $|x-a| > R$. In other words, we need to factor a 4 out of the absolute value bars in order to get the correct radius of convergence. Doing this gives,

$$8|x-2| < 1 \qquad \Rightarrow \qquad |x-2| < \frac{1}{8} \qquad \text{series converges}$$

$$8|x-2| > 1 \qquad \Rightarrow \qquad |x-2| > \frac{1}{8} \qquad \text{series diverges}$$

So, the radius of convergence for this power series is $R = \frac{1}{8}$.

Now, let's find the interval of convergence. Again, we'll first solve the inequality that gives convergence above.

$$-\frac{1}{8} < x - 2 < \frac{1}{8}$$
$$\frac{15}{8} < x < \frac{17}{8}$$

Now check the endpoints.

$x = \dfrac{15}{8}$:

The series here is,

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{15}{2} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty} \frac{2^n}{n}\frac{(-1)^n}{2^n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is the alternating harmonic series and we know that it converges.

$x = \dfrac{17}{8}$:

The series here is,

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{17}{2} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty} \frac{2^n}{n}\frac{1}{2^n}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

The interval of convergence for this power series is

$$\frac{15}{8} \le x < \frac{17}{8}$$

Example 3 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Solution
We'll start this example with the ratio test as we have for the previous ones.

$$L = \lim_{n\to\infty}\left|\frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)n!\,(2x+1)}{n!}\right|$$

$$= |2x+1|\lim_{n\to\infty}(n+1)$$

At this point we need to be careful. The limit is infinite, but there is that term with the x's in front of the limit. We'll have $L = \infty > 1$ provided $x \ne -\frac{1}{2}$.

Example 4 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$$

Solution
In this example the root test seems more appropriate. So,

$$L = \lim_{n\to\infty}\left|\frac{(x-6)^n}{n^n}\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left|\frac{x-6}{n}\right|$$

$$= |x-6|\lim_{n\to\infty}\frac{1}{n}$$

$$= 0$$

So, since $L = 0 < 1$ regardless of the value of x this power series will converge for every x.

In these cases we say that the radius of convergence is $R = \infty$ and interval of convergence is $-\infty < x < \infty$.

C- ***Alternating Series.***

alternating series is any series, $\sum a_n$, for which the series terms can be written in one of the following two forms.

$$a_n = (-1)^n b_n \qquad b_n \geq 0$$
$$a_n = (-1)^{n+1} b_n \qquad b_n \geq 0$$

There are many other ways to deal with the alternating sign, but they can all be written as one of the two forms above. For instance,

$$(-1)^{n+2} = (-1)^n (-1)^2 = (-1)^n$$
$$(-1)^{n-1} = (-1)^{n+1} (-1)^{-2} = (-1)^{n+1}$$

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim\limits_{n\to\infty} b_n = 0$ and,
2. $\{b_n\}$ is a decreasing sequence

the series $\sum a_n$ is convergent.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \qquad b_n = \frac{1}{n}$$

Now, all that we need to do is run through the two conditions in the test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Both conditions are met and so by the Alternating Series Test the series must converge. The series from the previous example is sometimes called the **Alternating Harmonic Series**. Also, the $(-1)^{n+1}$ could be $(-1)^n$ or any other form of alternating sign and we'd still call it an Alternating Harmonic Series.

Example 2 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5} = \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+5} \qquad \Rightarrow \qquad b_n = \frac{n^2}{n^2+5}$$

Let's check the conditions.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 0$$

So, the divergence test requires us to compute the following limit.

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5}$$

This limit can be somewhat tricky to evaluate. For a second let's consider the following,

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \left(\lim_{n\to\infty} (-1)^n\right)\left(\lim_{n\to\infty} \frac{n^2}{n^2+5}\right)$$

So, let's start with,

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \lim_{n\to\infty} \left[(-1)^n \frac{n^2}{n^2+5}\right]$$

Now, the second part of this clearly is going to 1 as $n \to \infty$ while the first part just alternates between 1 and -1. So, as $n \to \infty$ the terms are alternating between positive and negative values that are getting closer and closer to 1 and -1 respectively.

In order for limits to exist we know that the terms need to settle down to a single number and since these clearly don't this limit doesn't exist and so by the Divergence Test this series diverges.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-3} \sqrt{n}}{n+4}$$

Solution

$$b_n = \frac{\sqrt{n}}{n+4}$$

so let's check the conditions.

The first is easy enough to check.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$$

Let's start with the following function and its derivative.

$$f(x) = \frac{\sqrt{x}}{x+4} \qquad\qquad f'(x) = \frac{4-x}{2\sqrt{x}\,(x+4)^2}$$

Now, there are three critical points for this function, $x = -4$, $x = 0$, and $x = 4$. The first is outside the bound of our series so we won't need to worry about that one. Using the test points,

$$f'(1) = \frac{3}{50} \qquad\qquad f'(5) = -\frac{\sqrt{5}}{810}$$

and so we can see that the function in increasing on $0 \le x \le 4$ and decreasing on $x \ge 4$. Therefore, since $f(n) = b_n$ we know as well that the b_n are also increasing on $0 \le n \le 4$ and decreasing on $n \ge 4$.

The b_n are then eventually decreasing and so the second condition is met.

Both conditions are met and so by the Alternating Series Test the series must be converging.

Example 4 Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

Solution

The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,

$$\cos(n\pi) = (-1)^n$$

and so the series is really,

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \Rightarrow \qquad b_n = \frac{1}{\sqrt{n}}$$

Checking the two condition gives,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = b_{n+1}$$

The two conditions of the test are met and so by the Alternating Series Test the series is convergent.

5- ***Comparison Test .***

Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \ge 0$ for all n and $a_n \le b_n$ for all n. Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

consider the following series.

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}$$

$$\frac{1}{3^n + n} < \frac{1}{3^n}$$

Now,

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

is a geometric series and we know that since $|r| = \left|\frac{1}{3}\right| < 1$ the series will converge and its value will be,

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Now, if we go back to our original series and write down the partial sums we get,

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i}$$

Since all the terms are positive adding a new term will only make the number larger and so the sequence of partial sums must be an increasing sequence.

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i} < \sum_{i=0}^{n+1} \frac{1}{3^i + i} = s_{n+1}$$

Then since,

$$\frac{1}{3^n + n} < \frac{1}{3^n}$$

and because the terms in these two sequences are positive we can also say that,

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i} < \sum_{i=0}^{n} \frac{1}{3^i} < \sum_{i=0}^{\infty} \frac{1}{3^n} = \frac{3}{2} \qquad \Rightarrow \qquad s_n < \frac{3}{2}$$

So, the sequence of partial sums of our series is a convergent sequence. This means that the series itself,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}$$

is also convergent.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$$

Solution

Since the cosine term in the denominator doesn't get too large we can assume that the series terms will behave like,

$$\frac{n}{n^2} = \frac{1}{n}$$

Therefore,

$$\frac{n}{n^2 - \cos^2(n)} > \frac{n}{n^2} = \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (it's harmonic or the p-series test) by the Comparison Test our original series must also diverge.

Example 2 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$$

Solution

In this case the "+2" and the "+5" don't really add anything to the series and so the series terms should behave pretty much like

$$\frac{n^2}{n^4} = \frac{1}{n^2}$$

Let's take a look at the following series.

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4} + \sum_{n=1}^{\infty} \frac{2}{n^4}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^4}$$

As shown, we can write the series as a sum of two series and both of these series are convergent by the p-series test. Therefore, since each of these series are convergent we know that the sum,

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4}$$

is also a convergent series. Recall that the sum of two convergent series will also be convergent.

Now, since the terms of this series are larger than the terms of the original series we know that the original series must also be convergent by the Comparison Test.

6- ***Absolute Convergence***.

First, let's go back over the definition of absolute convergence.

Definition

A series $\sum a_n$ is called **absolutely convergent** if $\sum |a_n|$ is convergent. If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent we call the series **conditionally convergent**.

We also have the following fact about absolute convergence.

Fact

If $\sum a_n$ is absolutely convergent then it is also convergent.

Example 1 Determine if each of the following series are absolute convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$

(c) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

Solution

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

This is the alternating harmonic series and we saw in the last section that it is a convergent series so we don't need to check that here. So, let's see if it is an absolutely convergent series. To do this we'll need to check the convergence of.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

that it is divergent.

Therefore, this series is not absolutely convergent. It is however conditionally convergent since the series itself does converge.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$

In this case let's just check absolute convergence first since if it's absolutely convergent we won't need to bother checking convergence as we will get that for free.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+2}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This series is convergent by the p-series test and so the series is absolute convergent. Note that this does say as well that it's a convergent series.

(c) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^3}\right| = \sum_{n=1}^{\infty}\frac{|\sin n|}{n^3}$$

To do this we'll need to note that

$$-1 \le \sin n \le 1 \qquad \Rightarrow \qquad |\sin n| \le 1$$

and so we have,

$$\frac{|\sin n|}{n^3} \le \frac{1}{n^3}$$

Now we know that

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

converges by the p-series test and so by the Comparison Test we also know that

$$\sum_{n=1}^{\infty}\frac{|\sin n|}{n^3}$$

converges.

Therefore the original series is absolutely convergent (and hence convergent).

7- ***Ratio Test.***

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$$

Solution

With this first example let's be a little careful and make sure that we have everything down correctly. Here are the series terms a_n.

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Recall that to compute a_{n+1} all that we need to do is substitute $n+1$ for all the n's in a_n.

$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}\left((n+1)+1\right)} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$

Now, to define L we will use,

$$L = \lim_{n\to\infty}\left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

since this will be a little easier when dealing with fractions as we've got here. So,

$$\begin{aligned} L &= \lim_{n\to\infty}\left| \frac{(-10)^{n+1}}{4^{2n+3}(n+2)} \frac{4^{2n+1}(n+1)}{(-10)^n} \right| \\ &= \lim_{n\to\infty}\left| \frac{-10(n+1)}{4^2(n+2)} \right| \\ &= \frac{10}{16}\lim_{n\to\infty}\frac{n+1}{n+2} \\ &= \frac{10}{16} < 1 \end{aligned}$$

So, $L<1$ and so by the Ratio Test the series converges absolutely and hence will converge.

Example 2 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty}\frac{n!}{5^n}$$

Solution

Now that we've worked one in detail we won't go into quite the detail with the rest of these. Here is the limit.

$$L = \lim_{n\to\infty}\left| \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} \right| = \lim_{n\to\infty}\frac{(n+1)!}{5\,n!}$$

$$L = \lim_{n\to\infty}\frac{(n+1)\,n!}{5\,n!}$$

at which point we can cancel the $n!$ for the numerator an denominator to get,

$$L = \lim_{n\to\infty}\frac{(n+1)}{5} = \infty > 1$$

So, by the Ratio Test this series diverges.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$$

Solution

In this case be careful in dealing with the factorials.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2(n+1)-1)!} \frac{(2n-1)!}{n^2} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+1)!} \frac{(2n-1)!}{n^2} \right| \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n)(2n-1)!} \frac{(2n-1)!}{n^2} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n)(n^2)} \\ &= 0 < 1 \end{aligned}$$

So, by the Ratio Test this series converges absolutely and so converges.

Example 4 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1} n}$$

Solution

Do not mistake this for a geometric series. The n in the denominator means that this isn't a geometric series. So, let's compute the limit.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{9^{n+1}}{(-2)^{n+2}(n+1)} \frac{(-2)^{n+1} n}{9^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{9n}{(-2)(n+1)} \right| \\ &= \frac{9}{2} \lim_{n\to\infty} \frac{n}{n+1} \\ &= \frac{9}{2} > 1 \end{aligned}$$

Therefore, by the Ratio Test this series is divergent.

Example 5 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Solution
Let's first get L.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)^2+1} \frac{n^2+1}{(-1)^n} \right| = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = 1$$

So, as implied earlier we get $L = 1$ which means the ratio test is no good for determining the convergence of this series. We will need to resort to another test for this series. This series is an alternating series and so let's check the two conditions from that test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

$$b_n = \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = b_{n+1}$$

The two conditions are met and so by the Alternating Series Test this series is convergent. We'll leave it to you to verify this series is also absolutely convergent.
Example 6 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{n+2}{2n+7}$$

Solution
Here's the limit.

$$L = \lim_{n\to\infty} \left| \frac{n+3}{2(n+1)+7} \frac{2n+7}{n+2} \right| = \lim_{n\to\infty} \frac{(n+3)(2n+7)}{(2n+9)(n+2)} = 1$$

Again, the ratio test tells us nothing here. We can however, quickly use the divergence test on this. In fact that probably should have been our first choice on this one anyway.

$$\lim_{n\to\infty} \frac{n+2}{2n+7} = \frac{1}{2} \neq 0$$

By the Divergence Test this series is divergent.

8- <u>*Root Test.*</u>

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

4. if $L < 1$ the series is absolutely convergent (and hence convergent).
5. if $L > 1$ the series is divergent.
6. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Fact

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

Solution

There really isn't much to these problems other than computing the limit and then using the root test. Here is the limit for this problem.

$$L = \lim_{n\to\infty} \left| \frac{n^n}{3^{1+2n}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} = \infty > 1$$

So, by the Root Test this series is divergent.

Example 2 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \left(\frac{5n - 3n^3}{7n^3 + 2} \right)^n$$

Solution

Again, there isn't too much to this series.

$$L = \lim_{n\to\infty} \left| \left(\frac{5n - 3n^3}{7n^3 + 2} \right)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{5n - 3n^3}{7n^3 + 2} \right| = \left| \frac{-3}{7} \right| = \frac{3}{7} < 1$$

Therefore, by the Root Test this series converges absolutely and hence converges.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=3}^{\infty} \frac{(-12)^n}{n}$$

Solution

Here's the limit for this series.

$$L = \lim_{n\to\infty} \left| \frac{(-12)^n}{n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{12}{n^{\frac{1}{n}}} = \frac{12}{1} = 12 > 1$$

After using the fact from above we can see that the Root Test tells us that this series is divergent.

9- ***Taylor Series.***

So, for the time being, let's make two assumptions. First, let's assume that the function $f(x)$ does in fact have a power series representation about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

Next, we will need to assume that the function, $f(x)$, has derivatives of every order and that we can in fact find them all.

Now that we've assumed that a power series representation exists we need to determine what the coefficients, c_n, are. This is easier than it might at first appear to be. Let's first just evaluate everything at $x = a$. This gives,

$$f(a) = c_0$$

However, if we take the derivative of the function (and its power series) then plug in $x = a$ we get,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$
$$f'(a) = c_1$$

and we now know c_1.

Let's continue with this idea and find the second derivative.

$$f''(x) = 2c_2 + 3(2)c_3(x-a) + 4(3)c_4(x-a)^2 + \cdots$$
$$f''(a) = 2c_2$$

So, it looks like,

$$c_2 = \frac{f''(a)}{2}$$

Using the third derivative gives,

$$f'''(x) = 3(2)c_3 + 4(3)(2)c_4(x-a) + \cdots$$
$$f'''(a) = 3(2)c_3 \qquad \Rightarrow \qquad c_3 = \frac{f'''(a)}{3(2)}$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 4(3)(2)c_4 + 5(4)(3)(2)c_5(x-a) \cdots$$
$$f^{(4)}(a) = 4(3)(2)c_4 \qquad \Rightarrow \qquad c_4 = \frac{f^{(4)}(a)}{4(3)(2)}$$

Hopefully by this time you've seen the pattern here. It looks like, in general, we've got the following formula for the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This even works for *n=0* if you recall that $0! = 1$ and define $f^{(0)}(x) = f(x)$.

So, provided a power series representation for the function $f(x)$ about $x = a$ exists the **Taylor Series for** $f(x)$ **about** $x = a$ is,

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

If we use $a = 0$, so we are talking about the Taylor Series about $x = 0$, we call the series a **Maclaurin Series** for $f(x)$ or,

Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$
$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n^th^ degree Taylor polynomial of** $f(x)$ as,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Notice as well that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Next, the **remainder** is defined to be,

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is really just the *error* between the function $f(x)$ and the n^{th} degree Taylor polynomial for a given *n*.

With this definition note that we can then write the function as,

$$f(x) = T_n(x) + R_n(x)$$

Theorem

Suppose that $f(x) = T_n(x) + R_n(x)$. Then if,

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$ then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $|x-a| < R$.

Example 1 Find the Taylor Series for $f(x) = \mathbf{e}^x$ about $x = 0$.

Solution

To get a formula for $f^{(n)}(0)$ all we need to do is recognize that,

$$f^{(n)}(x) = \mathbf{e}^x \qquad n = 0, 1, 2, 3, \ldots$$

and so,

$$f^{(n)}(0) = \mathbf{e}^0 = 1 \qquad n = 0, 1, 2, 3, \ldots$$

Therefore, the Taylor series for $f(x) = \mathbf{e}^x$ about $x=0$ is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 2 Find the Taylor Series for $f(x) = \mathbf{e}^{-x}$ about $x = 0$.

Solution

Solution 1

As with the first example we'll need to get a formula for $f^{(n)}(0)$. However, unlike the first one we've got a little more work to do. Let's first take some derivatives and evaluate them at $x=0$.

$$\begin{array}{ll}
f^{(0)}(x) = \mathbf{e}^{-x} & f^{(0)}(0) = 1 \\
f^{(1)}(x) = -\mathbf{e}^{-x} & f^{(1)}(0) = -1 \\
f^{(2)}(x) = \mathbf{e}^{-x} & f^{(2)}(0) = 1 \\
f^{(3)}(x) = -\mathbf{e}^{-x} & f^{(3)}(0) = -1 \\
\vdots & \vdots \\
f^{(n)}(x) = (-1)^n \mathbf{e}^{-x} & f^{(n)}(0) = (-1)^n \qquad n = 0, 1, 2, 3
\end{array}$$

So, in this case we've got general formulas so all we need to do is plug these into the Taylor Series formula and be done with the problem.

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Solution 2

So, all we need to do is replace the x in the Taylor Series that we found in the first example with "$-x$".

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

This is a much shorter method of arriving at the same answer so don't forget about using previously computed series where possible (and allowed of course).

Example 3 Find the Taylor Series for $f(x) = x^4 \mathbf{e}^{-3x^2}$ about $x = 0$.

Solution

For this example we will take advantage of the fact that we already have a Taylor Series for $\mathbf{e}^x$ about $x = 0$. In this example, unlike the previous example, doing this directly would be significantly longer and more difficult.

$$\begin{aligned} x^4 \mathbf{e}^{-3x^2} &= x^4 \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} \\ &= x^4 \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+4}}{n!} \end{aligned}$$

To this point we've only looked at Taylor Series about $x = 0$ (also known as Maclaurin Series) so let's take a look at a Taylor Series that isn't about $x = 0$.

Example 4 Find the Taylor Series for $f(x) = \mathbf{e}^{-x}$ about $x = -4$.

Solution

Finding a general formula for $f^{(n)}(-4)$ is fairly simple.

$$f^{(n)}(x) = (-1)^n \mathbf{e}^{-x} \qquad\qquad f^{(n)}(-4) = (-1)^n \mathbf{e}^4$$

The Taylor Series is then,

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n \mathbf{e}^4}{n!} (x+4)^n$$

Example 5 Find the Taylor Series for $f(x)=\cos(x)$ about $x=0$.

Solution

First we'll need to take some derivatives of the function and evaluate them at *x=0*.

$$\begin{aligned}
f^{(0)}(x)&=\cos x & f^{(0)}(0)&=1\\
f^{(1)}(x)&=-\sin x & f^{(1)}(0)&=0\\
f^{(2)}(x)&=-\cos x & f^{(2)}(0)&=-1\\
f^{(3)}(x)&=\sin x & f^{(3)}(0)&=0\\
f^{(4)}(x)&=\cos x & f^{(4)}(0)&=1\\
f^{(5)}(x)&=-\sin x & f^{(5)}(0)&=0\\
f^{(6)}(x)&=-\cos x & f^{(6)}(0)&=-1
\end{aligned}$$

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n\\
&= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots\\
&= \underbrace{1}_{n=0}+\underbrace{0}_{n=1}\underbrace{-\frac{1}{2!}x^2}_{n=2}+\underbrace{0}_{n=3}\underbrace{+\frac{1}{4!}x^4}_{n=4}+\underbrace{0}_{n=5}\underbrace{-\frac{1}{6!}x^6}_{n=6}+\cdots
\end{aligned}$$

$$\cos x = \underbrace{1}_{n=0}\underbrace{-\frac{1}{2!}x^2}_{n=1}\underbrace{+\frac{1}{4!}x^4}_{n=2}\underbrace{-\frac{1}{6!}x^6}_{n=3}+\cdots$$

$$\cos x = \underbrace{1}_{n=0}\underbrace{-\frac{1}{2!}x^2}_{n=1}\underbrace{+\frac{1}{4!}x^4}_{n=2}\underbrace{-\frac{1}{6!}x^6}_{n=3}+\cdots$$

By renumbering the terms as we did we can actually come up with a general formula for the Taylor Series and here it is,

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

Example 6 Find the Taylor Series for $f(x) = \sin(x)$ about $x = 0$.

Solution

As with the last example we'll start off in the same manner.

$$\begin{aligned}
f^{(0)}(x) &= \sin x & \qquad f^{(0)}(0) &= 0 \\
f^{(1)}(x) &= \cos x & \qquad f^{(1)}(0) &= 1 \\
f^{(2)}(x) &= -\sin x & \qquad f^{(2)}(0) &= 0 \\
f^{(3)}(x) &= -\cos x & \qquad f^{(3)}(0) &= -1 \\
f^{(4)}(x) &= \sin x & \qquad f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \cos x & \qquad f^{(5)}(0) &= 1 \\
f^{(6)}(x) &= -\sin x & \qquad f^{(6)}(0) &= 0
\end{aligned}$$

So, we get a similar pattern for this one. Let's plug the numbers into the Taylor Series.

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots
\end{aligned}$$

So renumbering the terms as we did in the previous example we get the following Taylor Series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We really need to work another example or two in which $f(x)$ isn't about $x = 0$.

Example 7 Find the Taylor Series for $f(x) = \ln(x)$ about $x = 2$.

Solution

Here are the first few derivatives and the evaluations.

$$f^{(0)}(x) = \ln(x) \qquad f^{(0)}(2) = \ln 2$$

$$f^{(1)}(x) = \frac{1}{x} \qquad f^{(1)}(2) = \frac{1}{2}$$

$$f^{(2)}(x) = -\frac{1}{x^2} \qquad f^{(2)}(2) = -\frac{1}{2^2}$$

$$f^{(3)}(x) = \frac{2}{x^3} \qquad f^{(3)}(2) = \frac{2}{2^3}$$

$$f^{(4)}(x) = -\frac{2(3)}{x^4} \qquad f^{(4)}(2) = -\frac{2(3)}{2^4}$$

$$f^{(5)}(x) = \frac{2(3)(4)}{x^5} \qquad f^{(5)}(2) = \frac{2(3)(4)}{2^5}$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \qquad f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n} \qquad n = 1, 2, 3, \ldots$$

In order to plug this into the Taylor Series formula we'll need to strip out the $n = 0$ term first.

$$\begin{aligned}
\ln(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\ 2^n}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 2^n}(x-2)^n
\end{aligned}$$

Example 8 Find the Taylor Series for $f(x) = \frac{1}{x^2}$ about $x = -1$.

Solution

Again, here are the derivatives and evaluations.

$$f^{(0)}(x) = \frac{1}{x^2} \qquad f^{(0)}(-1) = \frac{1}{(-1)^2} = 1$$

$$f^{(1)}(x) = -\frac{2}{x^3} \qquad f^{(1)}(-1) = -\frac{2}{(-1)^3} = 2$$

$$f^{(2)}(x) = \frac{2(3)}{x^4} \qquad f^{(2)}(-1) = \frac{2(3)}{(-1)^4} = 2(3)$$

$$f^{(3)}(x) = -\frac{2(3)(4)}{x^5} \qquad f^{(3)}(-1) = -\frac{2(3)(4)}{(-1)^5} = 2(3)(4)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$f^{(n)}(x)=\frac{(-1)^n(n+1)!}{x^{n+2}} \qquad\qquad f^{(n)}(-1)=\frac{(-1)^n(n+1)!}{(-1)^{n+2}}=(n+1)!$$

Here is the Taylor Series for this function.

$$\begin{aligned}\frac{1}{x^2}&=\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n\\&=\sum_{n=0}^{\infty}\frac{(n+1)!}{n!}(x+1)^n\\&=\sum_{n=0}^{\infty}(n+1)(x+1)^n\end{aligned}$$

Example 9 Find the Taylor Series for $f(x)=x^3-10x^2+6$ about $x=3$.

Solution

Here are the derivatives for this problem.

$$\begin{aligned}f^{(0)}(x)&=x^3-10x^2+6 &\qquad f^{(0)}(3)&=-57\\f^{(1)}(x)&=3x^2-20x &\qquad f^{(1)}(3)&=-33\\f^{(2)}(x)&=6x-20 &\qquad f^{(2)}(3)&=-2\\f^{(3)}(x)&=6 &\qquad f^{(3)}(3)&=6\\f^{(n)}(x)&=0 &\qquad f^{(4)}(3)&=0 \qquad n\ge 4\end{aligned}$$

This Taylor series will terminate after $n=3$. This will always happen when we are finding the Taylor Series of a polynomial. Here is the Taylor Series for this one.

$$\begin{aligned}x^3-10x^2+6&=\sum_{n=0}^{\infty}\frac{f^{(n)}(3)}{n!}(x-3)^n\\&=f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3+0\\&=-57-33(x-3)-(x-3)^2+(x-3)^3\end{aligned}$$

Problems ***Sheet No.5***

10- Problems.

A- *Sequences.*

For problems 1 & 2 list the first 5 terms of the sequence.

1. $\left\{\dfrac{4n}{n^2-7}\right\}_{n=0}^{\infty}$

2. $\left\{\dfrac{(-1)^{n+1}}{2n+(-3)^n}\right\}_{n=2}^{\infty}$

For problems 3 – 6 determine if the given sequence converges or diverges. If it converges what is its limit?

3. $\left\{\dfrac{n^2-7n+3}{1+10n-4n^2}\right\}_{n=3}^{\infty}$

4. $\left\{\dfrac{(-1)^{n-2}n^2}{4+n^3}\right\}_{n=0}^{\infty}$

5. $\left\{\dfrac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}}\right\}_{n=1}^{\infty}$

6. $\left\{\dfrac{\ln(n+2)}{\ln(1+4n)}\right\}_{n=1}^{\infty}$

For each of the following problems determine if the sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

1. $\left\{\dfrac{1}{4n}\right\}_{n=1}^{\infty}$

2. $\left\{n(-1)^{n+2}\right\}_{n=0}^{\infty}$

3. $\left\{3^{-n}\right\}_{n=0}^{\infty}$

4. $\left\{\dfrac{2n^2-1}{n}\right\}_{n=2}^{\infty}$

5. $\left\{\dfrac{4-n}{2n+3}\right\}_{n=1}^{\infty}$

Problems ***Sheet No.5***

B- Series.

For problems 1 & 2 compute the first 3 terms in the sequence of partial sums for the given series.

1. $\sum_{n=1}^{\infty} n\,2^n$

2. $\sum_{n=3}^{\infty} \frac{2n}{n+2}$

For problems 3 & 4 assume that the n^{th} term in the sequence of partial sums for the series $\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

3. $s_n = \frac{5+8n^2}{2n-7n^2}$

4. $s_n = \frac{n^2}{5+2n}$

For problems 5 & 6 show that the series is divergent.

5. $\sum_{n=0}^{\infty} \frac{3n\,\mathbf{e}^n}{n^2+1}$

6. $\sum_{n=5}^{\infty} \frac{6+8n+9n^2}{3+2n+n^2}$

For each of the following series determine if the series converges or diverges. If the series converges give its value.

1. $\sum_{n=0}^{\infty} 3^{2+n}\,2^{1-3n}$

2. $\sum_{n=1}^{\infty} \frac{5}{6n}$

3. $\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}}$

4. $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$

Problems ***Sheet No.5***

C- Comparison Test.

For each of the following series determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1\right)^2$

2. $\sum_{n=4}^{\infty}\frac{n^2}{n^3-3}$

3. $\sum_{n=2}^{\infty}\frac{7}{n(n+1)}$

4. $\sum_{n=7}^{\infty}\frac{4}{n^2-2n-3}$

5. $\sum_{n=2}^{\infty}\frac{n-1}{\sqrt{n^6+1}}$

6. $\sum_{n=1}^{\infty}\frac{2n^3+7}{n^4\sin^2(n)}$

D- Absolute Convergence.

For each of the following series determine if they are absolutely convergent, conditionally convergent or divergent.

1. $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^3+1}$

2. $\sum_{n=1}^{\infty}\frac{(-1)^{n-3}}{\sqrt{n}}$

3. $\sum_{n=3}^{\infty}\frac{(-1)^{n+1}(n+1)}{n^3+1}$

Problems ***Sheet No.5***

E- Ratio Test.

For each of the following series determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$

2. $\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$

3. $\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$

4. $\sum_{n=3}^{\infty} \frac{\mathbf{e}^{4n}}{(n-2)!}$

F- Root Test.

For each of the following series determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$

2. $\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}$

3. $\sum_{n=4}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}}$

G- Power Series.

For each of the following power series determine the interval and radius of convergence.

1. $\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(4x-12)^n$

2. $\sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}}(2x+17)^n$

3. $\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)!}(x-2)^n$

4. $\sum_{n=0}^{\infty} \frac{4^{1+2n}}{5^{n+1}}(x+3)^n$

5. $\sum_{n=0}^{\infty} \frac{6^n}{n}(4x-1)^{n-1}$

Problems **_Sheet No.5_**

H- ***Taylor Series.***

For problems 1 & 2 use one of the Taylor Series derived in the notes to determine the Taylor Series for the given function.

1. $f(x) = \cos(4x)$ about $x = 0$

2. $f(x) = x^6 \mathbf{e}^{2x^3}$ about $x = 0$

For problem 3 – 6 find the Taylor Series for each of the following functions.

3. $f(x) = \mathbf{e}^{-6x}$ about $x = -4$

4. $f(x) = \ln(3 + 4x)$ about $x = 0$

5. $f(x) = \dfrac{7}{x^4}$ about $x = -3$